On spectral stability of the nonlinear Dirac equation

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November 14, 2012

Abstract

We study the point spectrum of the nonlinear Dirac equation linearized at one of the solitary wave solutions $\phi_\omega(x)e^{-i\omega t}$. We prove that, in any dimension, the linearized equation has no embedded eigenvalues in the part of the essential spectrum beyond the embedded thresholds (located at $\lambda = \pm i(m + |\omega|)$). We then prove that the birth of point eigenvalues with nonzero real part from the essential spectrum is only possible from the embedded eigenvalues, and therefore can not take place beyond the embedded thresholds. We also study the birth of point eigenvalues in the nonrelativistic limit, $\omega \to m$.

We apply our results to show that small amplitude solitary waves ($\omega \lesssim m$) of cubic nonlinear Dirac equation in one dimension are spectrally stable.

1 Introduction

While a lot is known about the nonlinear Schrödinger and Klein-Gordon equations (see e.g. the review [Str89]), there are still numerous open questions for systems with Hamiltonians that are not sign-definite, such as the Dirac-Maxwell system [Gro66, Wak66] and the nonlinear Dirac equation [Sol70]. There has been an enormous body of research devoted to the nonlinear Dirac equation, which we do not hope to cover comprehensively, only giving a very brief sketch. The existence of standing waves in the nonlinear Dirac equation was studied in [Sol70, CV86, Mer88, ES95]. Local and global well-posedness of the nonlinear Dirac equation was further addressed in [EV97] (semilinear Dirac equation in 3D) and in [MNNO05] (nonlinear Dirac equation in 3D). There are many results on the local and global well-posedness in 1D; we mention [ST10, MNT10, Can11, Pel11]. There were several attempts to study the stability of solitary waves of the nonlinear Dirac equation analyzing whether the energy functional is minimized with respect to dilations and other families of perturbations [Bog79, SV86, CKMS10, MQC$^+$12]. The spectrum of the linearization at solitary waves of the nonlinear Dirac equation in 1D was computed in [Chu07, BC12], suggesting the absence of eigenvalues with positive real part for linearizations at small amplitude solitary waves; we will say that such solitary waves are *linearly stable*, or *spectrally stable*. The numerical simulations of the evolution of perturbed solitary waves [MQC$^+$12] suggest that the small amplitude solitary waves in 1D nonlinear Dirac equation are *dynamically* stable (or *nonlinearly stable*). The asymptotic stability of small amplitude solitary waves in the external potential has been studied in [Bou06, Bou08, PS10]. The first approach to the translation-invariant case in 3D (based on the spectral stability assumptions) is in [BC11].

In the present paper, we study the spectral stability of solitary waves in the nonlinear Dirac equation. More precisely, we study the scenarios of the emergence of positive-real-part eigenvalues in the spectrum of the linearization at different solitary waves. We use our results to prove that the small amplitude solitary waves to the cubic nonlinear Dirac equation in 1D are spectrally stable; this is the first rigorous result of this type.

We consider the nonlinear Dirac equation in $\mathbb{R}^n$, $n \geq 1$:

$$\mathrm{i}\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi, \qquad \psi(x,t) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where N is even, $f \in C^\infty(\mathbb{R})$, $f(0) = 0$, and D_m is the free Dirac operator:

$$D_m = -\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m, \qquad m > 0, \tag{1.2}$$

$\boldsymbol{\alpha} = (\alpha^j)_{1\le j\le n}$, where α^j and β are self-adjoint $N\times N$ Dirac matrices. See Section 2 for the details. We are interested in the stability properties of the solitary wave solutions to (1.1):

$$\psi(x,t) = \phi_\omega(x)e^{-\mathrm{i}\omega t}, \qquad \phi_\omega \in H^\infty(\mathbb{R}^n, \mathbb{C}^N).$$

We consider the perturbation of a solitary wave, $(\phi_\omega(x) + \rho(x,t))e^{-\mathrm{i}\omega t}$, where ρ is a perturbation, and study the spectrum of the linearized equation on ρ. If the spectrum contains eigenvalues with positive real part, then the solitary wave is called *linearly unstable*, and one expects to be able to prove that this linear instability leads to the *orbital instability*, in the sense of [GSS87].

If the spectrum of the linearized equation is on the imaginary axis, we will say that the corresponding solitary wave is *spectrally stable*, or *linearly stable*. In this case, one hopes to prove the *asymptotic stability* of solitary waves using the dispersive estimates similarly to how this has been done for the nonlinear Schrödinger equation. First results in this direction are already appearing [BC11], with the assumptions on the spectrum of the linearized equation playing a crucial role. (Note that in the context of the nonlinear Dirac equation we do not know how to prove the *orbital stability* [GSS87] except via proving the asymptotic stability first.)

Since the isolated eigenvalues depend continuously on the perturbation, it is convenient to trace the behavior of "unstable" eigenvalues (eigenvalues with positive real part) for linearization at the solitary waves $\phi_\omega e^{-\mathrm{i}\omega t}$, considering ω as a parameter. For example, if one knows that solitary waves with ω in a certain interval are spectrally stable, one wants to know how and when the "unstable" eigenvalues may emerge from the imaginary axis. The emergence of unstable eigenvalues from $\lambda = 0$ is described by the Vakhitov-Kolokolov stability criterion [Com11]. In the present paper, we investigate the bifurcation of eigenvalues from the essential spectrum. We prove that such bifurcations are only possible from the part of the essential spectrum between the edge of the spectral gap and the embedded threshold: $\lambda \in \mathrm{i}\mathbb{R}$, $|\lambda| \in [m-|\omega|, m+|\omega|]$. We will show that such bifurcations, if they exist, can only originate from the embedded eigenvalues (with the exception of $\omega \in \{0; \pm m\}$, $\lambda = \pm\mathrm{i}(m-|\omega|)$).

Our approach to the spectral stability of solitary waves in the nonlinear Dirac equation is also applicable to the Dirac-Maxwell system. Let us mention that the local well-posedness of the Dirac-Maxwell system was proved in [Bou96], while the existence of standing waves in the Dirac-Maxwell system is proved in [EGS96] (for $\omega \in (-m, 0)$) and [Abe98] (for $\omega \in (-m, m)$). (For an overview of these results, see [ES02].) According to the nonrelativistic asymptotics ($\omega \gtrsim -m$) from [CS12], we expect that the solitary waves with ω sufficiently close to $-m$ are spectrally stable.

Another situation where our methods are applicable is the analysis of stability of gap solitons in nonlinear coupled-mode equations. Such systems appear in the context of photonic crystals [dSS94], where they describe counter-propagating light waves interacting with a linear grating in optical waveguides made of material with periodically changing refractive index [dSSS96, GWH01]. Coupled-mode systems also describe matter-wave Bose-Einstein condensates trapped in an optical lattice [PSK04]. The numerical analysis of the spectrum of the linearizations at the gap solitons is performed in [BPZ98, CP06]. The stability analysis of small-amplitude gap solitons based on the study of bifurcations from the embedded eigenvalues of the linear equation in the external potential is in [GW08].

Derrick's theorem

As a warm-up, let us consider the linear instability of stationary solutions to a nonlinear wave equation,

$$-\ddot{\psi} = -\Delta\psi + g(\psi), \qquad \psi = \psi(x,t) \in \mathbb{R}, \qquad x \in \mathbb{R}^n, \qquad n \ge 1, \qquad t \in \mathbb{R}. \tag{1.3}$$

We assume that the nonlinearity $g(\eta)$, $\eta \in \mathbb{R}$ is smooth. Equation (1.3) can be written as a Hamiltonian system $\dot{\pi} = -\delta_\psi E$, $\dot{\psi} = \delta_\pi E$, with the Hamiltonian $E(\psi,\pi) = \int_{\mathbb{R}^n}\left(\frac{\pi^2}{2} + \frac{|\nabla\psi|^2}{2} + G(\psi)\right)dx$, where $G(\eta) = \int_0^\eta g(\zeta)\,d\zeta$. There is a well-known result about non-existence of stable localized stationary solutions, known as *Derrick's theorem*:

Lemma 1.1 (Derrick's theorem [Der64]). *Equation* (1.3) *can not have stable, time-independent, localized solutions in three dimensions.*

Here is the argument from [Der64], which applies to dimensions $n \ge 3$. Denote

$$T(\theta) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\theta|^2\,dx, \qquad V(\theta) = \int_{\mathbb{R}^n} G(\theta)\,dx.$$

If $\psi(x,t)=\theta(x)$ is a localized stationary solution, so that $0=\dot\psi=\frac{\delta E}{\delta\pi}(\theta,0)$, $0=\dot\pi=-\frac{\delta E}{\delta\psi}(\theta,0)$, then for the family $\theta_\lambda(x)=\theta(x/\lambda)$, using the identities $T(\theta_\lambda)=\lambda^{n-2}T(\theta)$, $V(\theta_\lambda)=\lambda^n V(\theta)$, one has:

$$0=\Big\langle\frac{\delta E}{\delta\psi}(\theta,0),\frac{\partial\theta_\lambda}{\partial\lambda}\big|_{\lambda=1}\Big\rangle=\partial_\lambda|_{\lambda=1}E(\theta_\lambda,0)=(n-2)T(\theta)+nV(\theta),$$

and it follows that

$$\partial_\lambda^2|_{\lambda=1}E(\theta_\lambda)=(n-2)(n-3)T(\theta)+n(n-1)V(\theta)=-2(n-2)T(\theta),$$

which is negative as long as $n\ge 3$. That is, $\delta^2E<0$ for a variation corresponding to the uniform stretching, and the solution $\theta(x)$ from the physical point of view is to be unstable. We remark that the fact that $\partial_\lambda^2 E(\theta_\lambda)|_{\lambda=1}$ was not negative for $n=1$ and 2 does not prove that in these dimensions the localized stationary solutions are stable; it just means that a particular family of perturbations failed to catch the unstable direction. Let us modify Derrick's argument to show the linear instability of stationary solutions in any dimension.

Lemma 1.2 (Derrick's theorem for $n\ge 1$)**.** *For any $n\ge 1$, a smooth finite energy stationary solution $\theta\in H^\infty(\mathbb{R}^n)$ to the nonlinear wave equation* (1.3) *is linearly unstable.*

Proof. Since θ satisfies $-\Delta\theta+g(\theta)=0$, we also have $-\Delta\partial_{x_1}\theta+g'(\theta)\partial_{x_1}\theta=0$. Due to $\lim\limits_{|x|\to\infty}\theta(x)=0$, $\partial_{x_1}\theta$ vanishes somewhere. According to the minimum principle, there is a nowhere vanishing smooth function $\chi\in L^2(\mathbb{R}^n)$ ($\chi\in H^\infty(\mathbb{R}^n)$ due to Δ being elliptic) which corresponds to some smaller (hence negative) eigenvalue of $\mathfrak{l}=-\Delta+g'(\theta)$, $\mathfrak{l}\chi=-c^2\chi$, with $c>0$. Taking $\psi(x,t)=\theta(x)+r(x,t)$, we obtain the linearization at θ, $-\ddot r=-\mathfrak{l}r$, which we rewrite as

$$\partial_t\begin{bmatrix}r\\ s\end{bmatrix}=\begin{bmatrix}0&1\\ -\mathfrak{l}&0\end{bmatrix}\begin{bmatrix}r\\ s\end{bmatrix}.$$

The matrix in the right-hand side has eigenvectors $\begin{bmatrix}\chi\\ \pm c\chi\end{bmatrix}$, corresponding to the eigenvalues $\pm c\in\mathbb{R}$; thus, the solution θ is linearly unstable.

Let us also mention that $\partial_\tau^2|_{\tau=0}E(\theta+\tau\chi)<0$, showing that $\delta^2E(\theta)$ is not positive-definite. ☐

Remark 1.1. A more general result on linear instability of stationary solutions to (1.3) is proved in [KS07]. In particular, it is shown there that the linearization at a stationary solution may be spectrally stable when this particular stationary solution is not from H^1 (such examples exist in higher dimensions).

One can see that the linear stability analysis (Lemma 1.2) is more conclusive than the analysis of whether the energy functional is minimized under a particular families of perturbations or not (Lemma 1.1); in particular, we have just seen that, in the context of the nonlinear wave equation (1.3), the dilation perturbation fails to pick up the unstable directions in dimensions $n\le 2$. Moreover, for the nonlinear Dirac equation, it is easy to demonstrate that the solitary waves never correspond to the energy minima under the charge constraint, and, although we know that the energy is minimized at a solitary wave under some particular charge-preserving perturbations [Bog79, SV86, CKMS10, MQC$^+$12], it is not clear whether any conclusions could be drawn from this. This suggests that in order to have an insight about the stability of localized spinor solutions, one needs to perform the linear stability analysis.

Vakhitov-Kolokolov stability criterion for the nonlinear Schrödinger equation

For the nonlinear Schrödinger equation and several similar models, real eigenvalues could only emerge from the origin, and this emergence is controlled by the Vakhitov-Kolokolov stability condition [VK73]. Let us give the essence of the linear stability analysis on the example of the (generalized) nonlinear Schrödinger equation,

$$\mathrm{i}\partial_t\psi=-\frac{1}{2}\Delta\psi-f(|\psi|^2)\psi,\qquad \psi=\psi(x,t)\in\mathbb{C},\qquad x\in\mathbb{R}^n,\qquad n\ge 1,\qquad t\in\mathbb{R},\tag{1.4}$$

where $f(\eta)$ is a smooth function with $f(0)=0$. One can easily construct solitary wave solutions $\phi(x)e^{-\mathrm{i}\omega t}$, for some $\omega\in\mathbb{R}$ and $\phi\in H^1(\mathbb{R}^n)$: $\phi(x)$ satisfies the stationary equation $\omega\phi=-\frac{1}{2}\Delta\phi-f(\phi^2)\phi$, and can be chosen strictly positive, even, and monotonically decaying away from $x=0$. The value of ω can not exceed 0. We will only consider

the case $\omega < 0$. We use the Ansatz $\psi(x,t) = (\phi(x) + \rho(x,t))e^{-\mathrm{i}\omega t}$, with $\rho(x,t) \in \mathbb{C}$. The linearized equation on ρ is called the linearization at a solitary wave:

$$\partial_t \rho = \frac{1}{\mathrm{i}}\big(- \frac{1}{2}\Delta\rho - \omega\rho - f(\phi^2)\rho - 2f'(\phi^2)\phi^2 \operatorname{Re}\rho\big). \tag{1.5}$$

Remark 1.2. Because of the term with $\operatorname{Re}\rho$, the operator in the right-hand side is $\mathbb{R}$-linear but not $\mathbb{C}$-linear.

To study the spectrum of the operator in the right-hand side of (1.5), we first write it in the $\mathbb{C}$-linear form, considering its action onto $\boldsymbol{\rho}(x,t) = \begin{bmatrix}\operatorname{Re}\rho(x,t)\\ \operatorname{Im}\rho(x,t)\end{bmatrix}$:

$$\partial_t \boldsymbol{\rho} = \mathfrak{j}\mathfrak{l}\boldsymbol{\rho}, \qquad \boldsymbol{\rho}(x,t) = \begin{bmatrix}\operatorname{Re}\rho(x,t)\\ \operatorname{Im}\rho(x,t)\end{bmatrix}, \tag{1.6}$$

where

$$\mathfrak{j} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \qquad \mathfrak{l} = \begin{bmatrix}\mathfrak{l}_+ & 0\\ 0 & \mathfrak{l}_-\end{bmatrix}, \quad \text{with} \quad \mathfrak{l}_- = -\frac{1}{2}\Delta - \omega - f(\phi^2), \quad \mathfrak{l}_+ = \mathfrak{l}_- - 2f'(\phi^2)\phi^2. \tag{1.7}$$

If $\phi \in \mathcal{S}(\mathbb{R}^n)$, then by Weyl's theorem on the essential spectrum one has

$$\sigma_{\mathrm{ess}}(\mathfrak{l}_-) = \sigma_{\mathrm{ess}}(\mathfrak{l}_+) = [|\omega|, +\infty).$$

Lemma 1.3. $\sigma(\mathfrak{j}\mathfrak{l}) \subset \mathbb{R} \cup \mathrm{i}\mathbb{R}$.

Proof. We consider $(\mathfrak{j}\mathfrak{l})^2 = -\begin{bmatrix}\mathfrak{l}_-\mathfrak{l}_+ & 0\\ 0 & \mathfrak{l}_+\mathfrak{l}_-\end{bmatrix}$. Since $\mathfrak{l}_-$ is positive-definite ($\phi \in \ker \mathfrak{l}_-$, being nowhere zero, corresponds to the smallest eigenvalue), we can define the self-adjoint root of $\mathfrak{l}_-$; then

$$\sigma_d((\mathfrak{j}\mathfrak{l})^2)\backslash\{0\} = \sigma_d(\mathfrak{l}_-\mathfrak{l}_+)\backslash\{0\} = \sigma_d(\mathfrak{l}_+\mathfrak{l}_-)\backslash\{0\} = \sigma_d(\mathfrak{l}_-^{1/2}\mathfrak{l}_+\mathfrak{l}_-^{1/2})\backslash\{0\} \subset \mathbb{R},$$

with the inclusion due to $\mathfrak{l}_-^{1/2}\mathfrak{l}_+\mathfrak{l}_-^{1/2}$ being self-adjoint. Thus, any eigenvalue $\lambda \in \sigma_d(\mathfrak{j}\mathfrak{l})$ satisfies $\lambda^2 \in \mathbb{R}$. $\square$

Given the family of solitary waves, $\phi_\omega(x)e^{-\mathrm{i}\omega t}$, $\omega \in \mathcal{O} \subset \mathbb{R}$, we would like to know at which ω the eigenvalues of the linearized equation with $\operatorname{Re}\lambda > 0$ appear. Since $\lambda^2 \in \mathbb{R}$, such eigenvalues can only be located on the real axis, having bifurcated from $\lambda = 0$. One can check that $\lambda = 0$ belongs to the discrete spectrum of $\mathfrak{j}\mathfrak{l}$, with

$$\mathfrak{j}\mathfrak{l}\begin{bmatrix}0\\ \phi_\omega\end{bmatrix} = 0, \qquad \mathfrak{j}\mathfrak{l}\begin{bmatrix}-\partial_\omega\phi_\omega\\ 0\end{bmatrix} = \begin{bmatrix}0\\ \phi_\omega\end{bmatrix},$$

for all ω which correspond to solitary waves. Thus, if we will restrict our attention to functions which are spherically symmetric in x, the dimension of the generalized null space of $\mathfrak{j}\mathfrak{l}$ is at least two. Hence, the bifurcation follows the jump in the dimension of the generalized null space of $\mathfrak{j}\mathfrak{l}$. Such a jump happens at a particular value of ω if one can solve the equation $\mathfrak{j}\mathfrak{l}\alpha = \begin{bmatrix}\partial_\omega\phi_\omega\\ 0\end{bmatrix}$. This leads to the condition that $\begin{bmatrix}\partial_\omega\phi_\omega\\ 0\end{bmatrix}$ is orthogonal to the null space of the adjoint to $\mathfrak{j}\mathfrak{l}$, which contains the vector $\begin{bmatrix}\phi_\omega\\ 0\end{bmatrix}$; this results in $\langle\phi_\omega, \partial_\omega\phi_\omega\rangle = \partial_\omega\|\phi_\omega\|^2_{L^2}/2 = 0$. A slightly more careful analysis [CP03] based on construction of the moving frame in the generalized eigenspace of $\lambda = 0$ shows that there are two real eigenvalues $\pm\lambda \in \mathbb{R}$ that have emerged from $\lambda = 0$ when ω is such that $\partial_\omega\|\phi_\omega\|^2_{L^2}$ becomes positive, leading to a linear instability of the corresponding solitary wave. The opposite condition,

$$\partial_\omega\|\phi_\omega\|^2_{L^2} < 0, \tag{1.8}$$

is the Vakhitov-Kolokolov stability criterion which guarantees the absence of nonzero real eigenvalues for the nonlinear Schrödinger equation. It appeared in [VK73, CL82, Sha83, Wei86, GSS87] in relation to linear and orbital stability of solitary waves.

The above approach fails for the nonlinear Dirac equation since $\mathfrak{l}_-$ is no longer positive-definite. Now one no longer knows whether the eigenvalues are only on the real or imaginary axes, neither one knows whether (1.8) or its opposite is needed for stability. All we know is that (1.8) vanishes when the eigenvalues collide at $\lambda = 0$.

Our conclusions:

1. Point eigenvalues of the linearized Dirac equation may bifurcate (as ω changes) from the origin, when the dimension of the generalized null space jumps up (at the values of ω when $\partial_\omega \|\phi_\omega\|_{L^2} = 0$).
2. Since the spectrum of the linearization does not have to be a subset of $\mathbb{R}\cup\mathrm{i}\mathbb{R}$, there may also be point eigenvalues which bifurcate from the imaginary axis (either from the essential spectrum or from the collision of eigenvalues in the spectral gap) into the complex plane. We do not know particular examples of such behavior for the nonlinear Dirac equation.
3. Moreover, there may be point eigenvalues already present in the spectra of linearizations at arbitrarily small solitary waves. Formally, we could say that these eigenvalues bifurcate from the essential spectrum of the free Dirac equation, which can be considered as the linearization of the nonlinear Dirac equation at the zero solitary wave.

The first scenario has been studied in [Com11]. Here we will investigate the second and the third scenarios.

Here is the plan of the paper. The results are stated in Section 2. We consider the properties of solitary waves in Section 3. General properties of the linearization at solitary waves are considered in Section 4. Properties of embedded eigenvalues are in Section 5. Bifurcations of eigenvalues from the imaginary axis are considered in Sections 6, 7, and 8. The Hardy-type inequalities and Carleman-Berthier-Georgescu inequalities are considered in Appendices A and B.

2 Main results

Let Q be the operator of multiplication by x and $\langle Q\rangle$ be the operator of multiplication by $\sqrt{1+x^2}$.

For $u \in L^2(\mathbb{R}^n)$, we denote $\|u\| = \|u\|_{L^2}$. For $s,\ k \in \mathbb{R}$, we define

$$H^k_s(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n),\ \|u\|_{H^k_s} < \infty\right\}, \qquad \|u\|_{H^k_s} = \|\langle Q\rangle^s \langle\nabla\rangle^k u\|_{L^2}. \tag{2.1}$$

where if $k = 0$, we write L^2_s instead of H^0_s.

Let

$$D_m = -\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m, \qquad m > 0$$

be the free Dirac operator. Here $\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} = \sum_{\jmath=1}^n \alpha^\jmath \frac{\partial}{\partial x^\jmath}$, with $\alpha^\jmath$ and β being self-adjoint $N \times N$ Dirac matrices which satisfy

$$(\alpha^\jmath)^2 = \beta^2 = I_N, \qquad \alpha^\jmath\alpha^k + \alpha^k\alpha^\jmath = 2\delta_{\jmath k} I_N, \qquad \alpha^\jmath\beta + \beta\alpha^\jmath = 0, \qquad 1 \le \jmath, k \le n.$$

Above, I_N is the $N \times N$ identity matrix.

We consider the nonlinear Dirac equation (1.1),

$$\mathrm{i}\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi, \qquad \psi(x,t) \in \mathbb{C}^N, \quad x \in \mathbb{R}^n, \tag{2.2}$$

where $f \in C^\infty(\mathbb{R})$, $f(0) = 0$.

Assumption 1. There exists a nonempty open interval $\mathcal{O} \subseteq (-m, m)$ such that the equation

$$\omega u = D_m u - f(u^*\beta u)\beta u \tag{2.3}$$

admits a continuous family of solutions $\mathcal{O} \to H^\infty(\mathbb{R}^n)$, $\omega \mapsto \phi_\omega$.

Then $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ is a solitary wave solution to the nonlinear Dirac equation (2.2).

Properties of solitary wave solutions

Theorem 2.1. *Let $n \ge 1$, $f \in C^\infty(\mathbb{R})$, $f(0) = 0$.*

1. *Any solution ϕ_ω to* (2.3) *with some $\omega \in (-m, m)$, is such that then for any $\mu < \sqrt{m^2-\omega^2}$ one has $e^{\mu\langle Q\rangle}\phi_\omega \in L^\infty(\mathbb{R}^n, \mathbb{C}^N)$, where $\langle Q\rangle = \sqrt{1+x^2}$.*
2. *For $\omega \in \mathbb{R}\backslash[-m, m]$, there are no solitary wave solutions satisfying $\langle Q\rangle^{\frac14}\phi_\omega \in L^\infty(\mathbb{R}^n, \mathbb{C}^N)$.*

3. *Assume that $n \le 3$ and $k \in \mathbb{N}$. If $n = 3$, additionally assume that $k = 1$. If $f \in C^\infty(\mathbb{R})$ satisfies*

$$f(\eta) = \eta^k + o(\eta^k),$$

then there is $\omega_0 < m$ such that there are solitary wave solutions $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ to (2.2) with $\omega \in (\omega_0, m)$.

This theorem is proved in Section 3. In Section 3.2, we will also derive the nonrelativistic asymptotics of solitary waves as $\omega \to m$; this will allow us to make conclusions on the spectra of linearizations at solitary waves.

Properties of embedded eigenstates

Consider the solution to (2.2) in the form of the Ansatz $\psi(x,t) = (\phi_\omega(x) + \rho(x,t))e^{-\mathrm{i}\omega t}$, so that $\rho(x,t) \in \mathbb{C}^N$ is a small perturbation of the solitary wave. The linearization at the solitary wave $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ (the linearized equation on ρ) is given by

$$\partial_t \rho = \mathcal{J}\mathcal{L}(\omega)\rho, \tag{2.4}$$

where

$$\mathcal{J} = 1/i, \qquad \mathcal{L}(\omega) = D_m - \omega - f(\phi_\omega^*\beta\phi_\omega)\beta - 2f'(\phi_\omega^*\beta\phi_\omega)\beta\phi_\omega \operatorname{Re}(\phi_\omega^*\beta\ \cdot). \tag{2.5}$$

Remark 2.1. The operator $\mathcal{L}(\omega)$ is not $\mathbb{C}$-linear because of the term with $\operatorname{Re}(\phi_\omega^*\beta\ \cdot)$ (cf. Remark 1.2).

Let $\mathcal{O}$ be an open subset of $[-m, m]$ as in Assumption 1.

Theorem 2.2. 1. *If $\omega \in \mathcal{O}$ and $\lambda \in \sigma_p(\mathcal{J}\mathcal{L}(\omega)) \cap \mathrm{i}\mathbb{R}$, $|\lambda| < m + |\omega|$, then the corresponding eigenfunctions are exponentially decaying.*

2. *There are no embedded eigenvalues beyond the embedded thresholds:*

$$\sigma_p(\mathcal{J}\mathcal{L}(\omega)) \cap \mathrm{i}(\mathbb{R}\backslash[-m - |\omega|, m + |\omega|]) = \emptyset.$$

If $n = 1$, then additionally $\pm\mathrm{i}(m + |\omega|) \notin \sigma_p(\mathcal{J}\mathcal{L}(\omega))$.

We prove this theorem in Section 5.

Bifurcation of eigenvalues with nonzero real part

By Weyl's theorem on the essential spectrum (see Lemma 4.1 below), the essential spectrum of $\mathcal{J}\mathcal{L}(\omega)$ is purely imaginary; the discrete spectrum of is much more delicate. Our aim in this paper is to investigate the presence of point eigenvalues with positive real part. The presence of such eigenvalues leads to the linear instability of a particular solitary wave: certain perturbations start growing exponentially. As ω changes, such eigenvalues can bifurcate from the point spectrum on the imaginary axis or – possibly – even from the essential spectrum of $\mathcal{J}\mathcal{L}(\omega)$. We will show that the bifurcations of point eigenvalues from the essential spectrum into the half-planes with $\operatorname{Re}\lambda \ne 0$ are only possible from the embedded eigenvalues, and that there are no bifurcations beyond the embedded threshold. There are only two exceptions, which could occur when the edges of the continuous spectra meet: bifurcations from $\lambda = 0$ at $\omega = \pm m$ (as in [CGG12]) or bifurcations from $\pm m\mathrm{i}$ at $\omega = 0$ (as in [KS02]) do not have to start from embedded eigenvalues.

Theorem 2.3. *Let $n \ge 1$. Let $(\omega_j)_{j\in\mathbb{N}}$, $\omega_j \in \mathcal{O}$, be a Cauchy sequence, and let $\lambda_j \in \sigma_p(\mathcal{J}\mathcal{L}(\omega_j))$. Denote $\omega_b := \lim_{j\to\infty} \omega_j$, and let $\lambda_b \in \mathbb{C} \cup \{\infty\}$ be an accumulation point of $(\lambda_j)_{j\in\mathbb{N}}$. Then:*

1. *$\lambda_b \ne \infty$.*

2. *If $\lambda_b \in \mathrm{i}\mathbb{R}$, then $|\lambda_b| \le m + |\omega_b|$. If $n = 1$, then moreover $|\lambda_b| < m + |\omega_b|$.*

3. *If $\omega_b \in \mathcal{O}$ and $|\lambda_b| < m + |\omega_b|$, then $\lambda_b \in \sigma_p(\mathcal{J}\mathcal{L}(\omega_b))$, and there is a subsequence of eigenfunctions $(\zeta_j)_{j\in\mathbb{N}}$ corresponding to $\lambda_j \in \sigma_p(\mathcal{J}\mathcal{L}(\omega_j))$ which converges in L^2 to the eigenfunction ζ_b corresponding to $\lambda_b \in \sigma_p(\mathcal{J}\mathcal{L}(\omega_b))$.*

   Moreover, if there is a subsequence of $(\lambda_j)_{j\in\mathbb{N}}$ such that $\lambda_j \to \lambda_b$ and $\operatorname{Re}\lambda_j \ne 0$, then

$$\langle \zeta_b, \mathcal{L}(\omega_b)\zeta_b \rangle = 0.$$

4. *Define the zero order self-adjoint operator* $V(\omega) : L^2(\mathbb{R}^n, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ *by*

$$V(\omega) = \mathcal{L}(\omega) - D_m + \omega.$$

If $\omega_b = \pm m$ *and there is* $s > 1/2$ *such that*

$$\lim_{j\to\infty} \|\langle Q\rangle^{2s} V(\omega_j)\|_{L^\infty(\mathbb{R}^n, \mathrm{End}(\mathbb{C}^N))} = 0,$$

then $\lambda_b \in \{0; \pm 2m\mathrm{i}\}$.

If additionally $\mathrm{Re}\,\lambda_j \neq 0$, *then* $\lambda_b = 0$ *and moreover* $\lambda_j = O(m - |\omega_j|)$.

This theorem will be proved in Sections 6 and 7.

We are going to relate the families of eigenvalues of the linearized nonlinear Dirac equation bifurcating from $\lambda = 0$ with the eigenvalues of the linearized nonlinear Schrödinger equation. Let $n \le 3$. Let $u_k(y)$, $k \in \mathbb{N}$ ($k = 1$ if $n = 3$) be a strictly positive spherically symmetric function to the equation

$$-\frac{1}{2m} u_k = -\frac{1}{2m}\Delta u_k - |u_k|^{2k} u_k, \qquad u_k \in \mathcal{S}(\mathbb{R}^n), \tag{2.6}$$

so that $u_k(x)e^{-\mathrm{i}\omega t}$ with $\omega = -\frac{1}{2m}$ is a solitary wave solution to the nonlinear Schrödinger equation

$$\mathrm{i}\dot\psi = -\frac{1}{2m}\Delta\psi - |\psi|^{2k}\psi.$$

By (1.7), the linearization at this solitary wave is given by

$$\partial_t \boldsymbol{\rho} = \mathbf{j}\mathfrak{l}\boldsymbol{\rho},$$

where $\mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\mathfrak{l} = \begin{bmatrix} \mathfrak{l}_+ & 0 \\ 0 & \mathfrak{l}_- \end{bmatrix}$, and

$$\mathfrak{l}_- = \frac{1}{2m} - \frac{\Delta}{2m} - u_k^{2k}, \qquad \mathfrak{l}_+ = \frac{1}{2m} - \frac{\Delta}{2m} - (2k+1)u_k^{2k}, \tag{2.7}$$

where u_k is a strictly positive spherically-symmetric solution to (2.6).

Remark 2.2. By [BL83, Example 1], equation (2.6) in $\mathbb{R}^n$ with $k \in \mathbb{N}$ has nontrivial solutions if and only if $n \le 3$; if $n = 3$, one additionally requires $k = 1$.

Theorem 2.4. *Let* $n \le 3$, *and let* $f \in C^\infty(\mathbb{R})$, $f(\eta) = \eta^k + o(\eta^k)$, $k \in \mathbb{N}$. *If* $n = 3$, *further assume that* $k = 1$. *Let* $\phi_\omega e^{-\mathrm{i}\omega t}$ *be solitary waves from Lemma 3.5 (see below). Let* $\omega_j \to m$, *and let* $\lambda_j \in \sigma_p(\mathcal{J}\mathcal{L}(\omega_j))$, $\lambda_j = O(m^2 - \omega_j^2)$. *Assume that the operator* $\mathfrak{l}_-$ *introduced in* (2.7) *satisfies* $\sigma_p(\mathfrak{l}_-) = \{0\}$.

1. *Assume that* $\Lambda = 0$ *is an eigenvalue of* $\mathbf{j}\mathfrak{l}$ *of algebraic multiplicity* $2 + 2n$ *(either* $n = 1$, $k \neq 2$; *or* $n = 2$, $k > 1$; *or* $n = 3$*). Then* $\lambda_j \neq 0$ *for* $j \in \mathbb{N}$ *implies that*

$$\Lambda_b := \lim_{j\to\infty} \frac{\lambda_j}{m^2 - \omega_j^2} \neq 0.$$

2. $\Lambda_b \in \sigma_p(\mathbf{j}\mathfrak{l})$ *or* $\Lambda_b \in \sigma_p(\mathfrak{l}_-)$.

3. *If* $\mathrm{Re}\,\lambda_j \neq 0$, $j \in \mathbb{N}$, *then* Λ_b *has zero Krein signature: a corresponding eigenvector* $\boldsymbol{z} \in L^2(\mathbb{R}^n)$ *satisfies*

$$\langle \boldsymbol{z}, \mathfrak{l}\boldsymbol{z}\rangle = 0.$$

Remark 2.3. By [CGNT08], the condition $\sigma_p(\mathfrak{l}_-) = \{0\}$ is satisfied in the case $n = 1$, $k = 1$ or $k = 2$, and also in the case $n = 2$, $k = 1$. In the cases $n = 1$, $k \ge 3$ or $n = 2$, $k \ge 2$, or $n = 3$, $k = 1$ the small solitary waves are already known to be unstable due to the presence of positive eigenvalue in the spectrum of the linearized equation; see [CGG12].

Remark 2.4. The algebraic multiplicity of $\Lambda = 0 \in \sigma_p(\mathbf{j}\mathfrak{l})$ is at least $2n + 2$. Indeed, since $\ker \mathfrak{l}_- = \mathrm{Span}\{u_k\}$, $\ker \mathfrak{l}_+ = \mathrm{Span}\{\partial_j u_k \,;\, 1 \le j \le n\}$, $\dim\ker \mathbf{j}\mathfrak{l} = n + 1$. Moreover, each of these null eigenvectors, being orthogonal to the kernel of $(\mathbf{j}\mathfrak{l})^* = -\mathfrak{l}\mathbf{j}$, has an adjoint eigenvector.

Spectral stability of solitary waves of the nonlinear Dirac equation in 1D

We use the above results to prove the following spectral stability result.

Theorem 2.5. *Let $f \in C^\infty(\mathbb{R})$, $f(0) = 0$, $f'(0) > 0$. There is $\omega_0 \in (0, m)$ such that there is a family of solitary wave solutions $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ to the nonlinear Dirac equation,*

$$i\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi, \qquad \psi(x,t) \in \mathbb{C}^2, \qquad x \in \mathbb{R}, \tag{2.8}$$

and for each $\omega \in (\omega_0, m)$ the corresponding solitary wave is spectrally stable.

Above, $D_m = -i\alpha\frac{\partial}{\partial x} + \beta m$, where α, β are self-adjoint matrices and satisfy $\alpha^2 = \beta^2 = I_2$, $\{\alpha, \beta\} = 0$.

Proof. We consider the family of solitary wave solutions $\phi_\omega e^{-\mathrm{i}\omega t}$ which is described in Lemma 3.5 below.

Remark 2.5. In 1D, this family is unique, in the sense that for each $\omega \in (-m, m) \setminus \{0\}$ there is at most one solitary wave (see e.g. [BC12]), modulo the translations in x and the $\mathbf{U}(1)$-invariance.

We assume that there is a family of eigenvalues $\lambda_j \in \sigma_p(\mathbf{JL}(\omega))$, with $\operatorname{Re}\lambda_j \neq 0$. Then, by Theorem 2.2, $\lambda_j \to 0$ and moreover $\Lambda_j := \frac{\lambda_j}{m^2-\omega_j^2} \to \Lambda_b \in \sigma(\mathbf{j}\mathbf{l})$, where $\mathbf{j}\mathbf{l}$ is the linearization of the cubic NLS in 1D, ((1.4)) with $n = 1$ and $f(\eta) = \eta + o(\eta)$). By Theorem 2.4 *(2)*, $\Lambda_b := \lim_{\omega_j\to m}\frac{\lambda_j}{m^2-\omega_j^2}$ belongs to the point spectrum of the cubic NLS linearized at a solitary wave; by [CGNT08, Fig. 1, $p = 3$], this spectrum consists of $\lambda = 0$ only (note that there could be no embedded eigenvalues in 1D due to the asymptotics of the Jost solutions), hence we must have $\Lambda_b = 0$.

By Theorem 2.4 *(1)*, since the generalized null space of the linearization of the cubic NLS is four-dimensional, $\Lambda_j = 0$ for all but finitely many j; thus, there is no sequence $(\lambda_j)_{j\in\mathbb{N}}$ with the above properties. □

Remark 2.6. A slightly more careful analysis shows that small amplitude solitary wave solutions to the Dirac equation in 1D with quintic nonlinearity and in 2D with cubic nonlinearity are spectrally stable. (In other cases, small amplitude solitary waves are linearly unstable [CGG12].)

Remark 2.7. While we try to exclude the bifurcations of nonzero-real-part eigenvalues from the essential spectrum, there is a possibility that, as ω changes, purely imaginary point eigenvalues bifurcate from the edges of the essential spectrum into the spectral gap (so that $\operatorname{Re}\lambda = 0$) even if when there are no embedded eigenvalues at the edges. This was noticed numerically in e.g. [BC12]. Then a pair of purely imaginary eigenvalues could either collide at $\lambda = 0$ and turn into a pair of one positive and one negative eigenvalues (this collision is characterized by the Vakhitov-Kolokolov condition $\partial_\omega\|\phi_\omega\|^2 = 0$; see e.g. [Com11]), or two pairs of purely imaginary eigenvalues could collide in the gap but away from $\lambda = 0$, producing a two pairs of eigenvalues with nonzero real parts.

3 Solitary wave solutions

3.1 General properties

Lemma 3.1. *Let $n \geq 1$. Let $\phi_\omega \in L^2(\mathbb{R}^n, \mathbb{C}^N)$, $\omega \in (-m, m)$ be a solution to* (2.3)*. Then for any $\mu < \sqrt{m^2 - \omega^2}$ one has*

$$e^{\mu\langle Q\rangle}\phi_\omega \in L^\infty(\mathbb{R}^n, \mathbb{C}^N).$$

As before, $\langle Q\rangle$ is the operator of multiplication by $\sqrt{1 + x^2}$.

Proof. For the sake of completeness, we choose to provide a proof of the above lemma. We will use the Combes-Thomas method, see [His00]. The solitary wave profile ϕ_ω satisfies

$$\omega\phi_\omega = D_m\phi_\omega - f(\phi_\omega^*\beta\phi_\omega)\beta\phi_\omega. \tag{3.1}$$

Pick $\mu \in (0, \sqrt{m^2 - \omega^2})$ and let $\epsilon \in (0, \sqrt{m^2 - \omega^2} - \mu)$. Due to the Assumption 1, $\phi_\omega(x)$ are smooth functions of x, tending to 0 as $|x| \to \infty$; so is $f(\phi_\omega^*\beta\phi_\omega)$. Therefore, we can write

$$f(\phi_\omega^*\beta\phi_\omega) = F_0(x, \omega) + F_1(x, \omega), \qquad x \in \mathbb{R}^n,$$

where F_0 and F_1 are smooth and F_0 has compact support, while $\sup_{x\in\mathbb{R}^n,\,\omega\in\mathcal{I}}|F_1(x,\omega)|\le\epsilon$. For any $\varphi\in C^2(\mathbb{R}^n)$, we have:

$$e^{\varphi}(D_m-\omega-F_1\beta)e^{-\varphi}e^{\varphi}\phi_\omega=F_0\beta e^{\varphi}\phi_\omega,$$

or

$$(D_m-\omega-F_1\beta+D_0\varphi)e^{\varphi}\phi_\omega=F_0\beta e^{\varphi}\phi_\omega. \tag{3.2}$$

We take $\varphi(x)=\mu\rho(x)$, with $\rho(x)=\langle x\rangle:=\sqrt{1+x^2}$, so that $\lim\limits_{|x|\to\infty}|\nabla\rho(x)|=1$, $\|\nabla\rho\|_{L^\infty}=1$. By (3.2),

$$e^{\mu\rho}\phi_\omega=(D_m-\omega-F_1\beta+\mu D_0\rho)^{-1}F_0\beta e^{\mu\rho}\phi_\omega.$$

The invertibility of $D_m-\omega-F_1\beta+\mu D_0\rho$ follows from the condition on μ and the estimate

$$\|(D_m-\omega-F_1\beta)u\|\ge\operatorname{dist}(\omega,\pm m)-\epsilon,$$

leading to

$$\|e^{\mu\rho}\phi_\omega\|\le(\sqrt{m^2-\omega^2}-\epsilon-\mu\|\nabla\rho\|_{L^\infty})^{-1}\|F_0\beta e^{\mu\rho}\|_{L^\infty}\|\phi_\omega\|=\frac{\|F_0e^{\mu\rho}\|_{L^\infty}\|\phi_\omega\|}{\sqrt{m^2-\omega^2}-\epsilon-\mu}.$$

Since $e^{\mu\rho}\phi_\omega$ satisfies the elliptic equation (3.2), boundedness of $e^{\mu\rho}\phi_\omega$ in $L^2(\mathbb{R}^n,\mathbb{C}^N)$ leads to its boundedness in $H^k(\mathbb{R}^n,\mathbb{C}^N)$, for any $k\in\mathbb{N}$. By the Sobolev embedding theorem, $\|e^{\mu\rho}\phi_\omega\|_{L^\infty}$ is bounded. □

For solitary waves associated to embedded eigenvalues the situation is, in many respects, different. We can first notice that with Hardy type estimates, we have

Lemma 3.2. *Let $n\ge1$. Let $\omega\in\mathbb{R}\setminus[-m,m]$. If a solitary wave $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ associated to ω is such that*

$$\langle Q\rangle^{1/2+\varepsilon}\phi_\omega\in L^\infty(\mathbb{R}^n,\mathbb{C}^N)$$

for some $\varepsilon>0$, then ϕ_ω decays faster than algebraically in x. That is, for any $K>0$,

$$\langle Q\rangle^{K}\phi_\omega\in L^\infty(\mathbb{R}^n,\mathbb{C}^N).$$

Proof. Since the nonlinearity $f\in C^\infty(\mathbb{R})$ from (2.2) satisfies $f(0)=0$, there is $g\in C^\infty(\mathbb{R})$ such that $f(\eta)=\eta g(\eta)$. Then, from Lemma A.5, for any $s>-1/2$ there is a constant c such that

$$\|\langle Q\rangle^s\phi_\omega\|_{H^1}\le c\|\langle Q\rangle^{s+1}(D_m-\lambda)\phi_\omega\|\le C\|g(\psi^*\beta\psi)(\langle Q\rangle^{1/2+\varepsilon}\psi^*\beta\langle Q\rangle^{1/2+\varepsilon}\psi)\beta\langle Q\rangle^{s-2\varepsilon}\phi_\omega\|,$$

which bootstraps immediately. □

The situation is even more dramatic if one applies the Carleman-Berthier-Georgescu inequalities from Appendix B.

Lemma 3.3. *There are no solitary wave solutions $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ with $\omega\in\mathbb{R}\setminus[-m,m]$ such that*

$$\langle Q\rangle^{1/4}\phi_\omega\in L^\infty(\mathbb{R}^n,\mathbb{C}^N). \tag{3.3}$$

Proof. The proof goes in three steps.

1. Let $\omega\in\mathbb{R}$, $|\omega|>m$. Denote $\Omega_R=\{x\in\mathbb{R}^n\ ;\ |x|>R\}$. We have the following: There exists $R_0\ge0$, $\tau_0>0$ and $C(R_0)>0$ such that for any $R\ge R_0$, for any $u\in H^1_0(\Omega_R,\mathbb{C}^N)$, and for any $\tau\ge\tau_0$ there is the inequality

$$\|e^{\tau|Q|}u\|_{L^2(\Omega_R,\mathbb{C}^N)}\le\frac{C(R_0)}{\tau}\|r^{1/2}e^{\tau|Q|}(D_m-f(\phi_\omega^*\beta\phi_\omega)-\omega)u\|_{L^2(\Omega_R,\mathbb{C}^N)}.$$

Indeed, by Lemma B.4, there are $C,R<\infty$ such that for any $u\in H^1_c(\Omega_R,\mathbb{C}^N)$ and $\tau\ge1$

$$\tau\|e^{\tau r}u\|\le C\|r^{1/2}e^{\tau r}(D_m-\lambda)u\|,$$

and thus

$$\tau\|e^{\tau r}u\|_{L^2(\Omega_R,\mathbb{C}^N)} \le \frac{C}{\tau}\|r^{1/2}e^{\tau r}(D_m - f(\phi_\omega^*\beta\phi_\omega)-\omega)u\|_{L^2(\Omega_R,\mathbb{C}^N)} + \frac{C}{\tau}\|e^{\tau r}r^{1/2}f(\phi_\omega^*\beta\phi_\omega)u\|_{L^2(\Omega_R,\mathbb{C}^N)}.$$

So that from the assumption, we obtain the claim for sufficiently large τ.

2. Now we can prove that a solitary wave ϕ_ω with $|\omega| > m$ which satisfies (3.3) is smooth and with compact support. Indeed, the smoothness follows from Assumption 1. Then consider $v_j = \eta_j\phi$ with $\eta_j := \eta(\cdot/j)$, where η is smooth and satisfies $0 \le \eta \le 1$, identically equals 1 outside the ball of radius 2, and identically equals zero inside the ball of radius 1. From the previous step, it follows that, for $\tau > 1$ and for $j > R_0$ sufficiently large,

$$\|e^{\tau|Q|}v_j\|_{L^2(\Omega_R,\mathbb{C}^N)} \le \frac{2C(R_0)}{\tau}\||Q|^{1/2}e^{\tau|Q|}(D_m - f(\phi_\omega^*\beta\phi_\omega)-\omega)v_j\|_{L^2(\Omega_R,\mathbb{C}^N)}.$$

Therefore, $\|e^{\tau|Q|}v_j\|_{L^2(\Omega_R,\mathbb{C}^N)} \le \frac{2C(R_0)}{\tau}\||Q|^{1/2}e^{\tau|Q|}\alpha\cdot(\nabla\eta_j)\phi_\omega\|_{L^2(\Omega_R,\mathbb{C}^N)}$, which implies that

$$e^{\tau 3k}\|\phi_\omega\|_{L^2(\Omega_{3k},\mathbb{C}^N)} \le \ \mathrm{const}\,\frac{2C(R_0)}{\tau}e^{\tau 2k}\|\phi_\omega\|_{L^2(\Omega_k\cap B_{2k},\mathbb{C}^N)},$$

where

$$B_R = \{x\in\mathbb{R}^n\ ;\ |x| < R\}.$$

Since τ could be arbitrarily large, one concludes that ϕ_ω is identically zero outside of the ball B_{3k}.

3. Now we need the unique continuation principle for the Dirac operator [BG87].

Lemma 3.4. *Let Ω an open connected subset of $\mathbb{R}^n$. Let $v:\ \Omega\to\mathbb{R}^+$ be in $L^5_{\mathrm{loc}}(\Omega)$. If $\psi\in H^1_{\mathrm{loc}}(\Omega)$ satisfies*

$$|(D_0\psi)(x)| \le v(x)|\psi(x)| \quad \textit{a.e. in } \Omega$$

and $\psi(x) = 0$ in an open non-empty subset of Ω, then $\psi\equiv 0$.

We refer to [BG87, Appendix] for the proof. Although it is written in dimension $n = 3$, the key part [BG87, Appendix, Lemma 1] is true in any dimension.

From Lemma (3.4), the solitary wave ϕ_ω is identically zero. □

Remark 3.1. The statements of the above lemmas can be improved with the stronger assumptions on f. For instance, if $f(\eta) = O(|\eta|^k)$, then there are no solitary wave solutions $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ with $\omega\in\mathbb{R}\setminus[-m,m]$ such that

$$\langle Q\rangle^{1/2k}\phi_\omega \in L^\infty(\mathbb{R}^n,\mathbb{C}^N).$$

Lemmas 3.1 and 3.3 complete the proof of Theorem 2.1.

3.2 Solitary waves in the nonrelativistic limit

In this section, we will study asymptotics of solitary waves in the nonrelativistic limit $\omega\to m$. The proof of the following lemma is given in [CGG12].

Lemma 3.5. *Let $n \le 3$. Let*

$$f\in C^\infty(\mathbb{R}), \qquad f(\eta) = \eta^k + o(\eta^k), \qquad k\in\mathbb{N}. \tag{3.4}$$

If $n = 3$, additionally assume that $k = 1$. There is $\omega_0 < m$, dependent on n and f, such that for $\omega\in(\omega_0,m)$ there are solitary wave solutions $\phi_\omega(x)e^{-\mathrm{i}\omega t}$, with ϕ_ω satisfying

$$\omega\phi_\omega = D_m\phi - f(\phi_\omega^*\beta\phi_\omega)\beta\phi_\omega. \tag{3.5}$$

Moreover, introducing the projections onto the "particle" and "antiparticle" components,

$$\Pi_P = \frac{1}{2}(1+\beta), \qquad \Pi_A = \frac{1}{2}(1-\beta); \qquad \phi_P(x) = \Pi_P\phi(x), \qquad \phi_A(x) = \Pi_A\phi(x),$$

we have for $\epsilon=\sqrt{m^2-\omega^2}$

$$\phi_P(x)=\epsilon^{\frac{1}{k}}\hat{\varPhi}_P(\epsilon x)+O_{H^2}(\epsilon^{2+\frac{1}{k}}),\qquad \phi_A(x)=\epsilon^{1+\frac{1}{k}}\hat{\varPhi}_A(\epsilon x)+O_{H^2}(\epsilon^{3+\frac{1}{k}}), \tag{3.6}$$

where $\hat{\varPhi}_P(y)$*,* $\hat{\varPhi}_A(y)$ *satisfy*

$$-\frac{1}{2m}\hat{\varPhi}_P=-\frac{1}{2m}\Delta\hat{\varPhi}_P-|\hat{\varPhi}_P|^2k\hat{\varPhi}_P,\qquad \hat{\varPhi}_A=\frac{1}{2m}(-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_y)\hat{\varPhi}_P. \tag{3.7}$$

One can choose

$$\hat{\varPhi}_P(y)=\boldsymbol{n}u_k(y), \tag{3.8}$$

where $\boldsymbol{n}\in\mathbb{C}^N$*,* $\|\boldsymbol{n}\|=1$*, and* $u_k\in\mathcal{S}(\mathbb{R}^n)$ *is a strictly positive spherically symmetric solution to* (2.6)*.*

Remark 3.2. As we mentioned in Remark 2.2, equation (2.6) in $\mathbb{R}^n$ with $k\in\mathbb{N}$ has nontrivial solutions if and only if $n\le 3$; if $n=3$, one additionally requires $k=1$. Absence of corresponding solutions to (3.7) does not allow us to construct small amplitude solitary wave solutions (in the nonrelativistic limit $\omega\to m$) to the nonlinear Dirac equation in $\mathbb{R}^3$ with $k>2$ and in $\mathbb{R}^n$, $n>3$, with any $k\in\mathbb{N}$.

4 Linearization at a solitary wave

4.1 Essential spectrum of the linearization operator

Denote

$$\mathcal{L}_-(\omega)=D_m-\omega-f(\phi_\omega^*\beta\phi_\omega)\beta,\qquad \mathscr{X}:=\mathcal{D}(\mathcal{L}_-(\omega))=H^1(\mathbb{R}^n,\mathbb{C}^N). \tag{4.1}$$

Introduce

$$\boldsymbol{\alpha}^\jmath=\begin{bmatrix}\mathrm{i}\,\mathrm{Im}\,\alpha^\jmath & \mathrm{i}\,\mathrm{Re}\,\alpha^\jmath\\ -\mathrm{i}\,\mathrm{Re}\,\alpha^\jmath & \mathrm{i}\,\mathrm{Im}\,\alpha^\jmath\end{bmatrix},\quad 1\le\jmath\le n;\qquad \boldsymbol{\beta}=\begin{bmatrix}\mathrm{Re}\,\beta & -\mathrm{Im}\,\beta\\ \mathrm{Im}\,\beta & \mathrm{Re}\,\beta\end{bmatrix},\qquad \mathbf{J}=\begin{bmatrix}0 & I_N\\ -I_N & 0\end{bmatrix}. \tag{4.2}$$

Let

$$\boldsymbol{\phi}_\omega(x)=\begin{bmatrix}\mathrm{Re}\,\phi_\omega(x)\\ \mathrm{Im}\,\phi_\omega(x)\end{bmatrix}\in\mathbb{R}^{2N}.$$

Denote

$$\mathbf{L}_-(\omega)=\mathbf{D}_m-\omega-f(\boldsymbol{\phi}_\omega^*\boldsymbol{\beta}\boldsymbol{\phi}_\omega)\boldsymbol{\beta},\qquad \mathbf{L}(\omega)=\mathbf{L}_-(\omega)-2f'(\boldsymbol{\phi}_\omega^*\boldsymbol{\beta}\boldsymbol{\phi}_\omega)(\boldsymbol{\phi}_\omega^*\boldsymbol{\beta}\ \cdot)\boldsymbol{\beta}\boldsymbol{\phi}_\omega. \tag{4.3}$$

Above, $\mathbf{D}_m=-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+m\boldsymbol{\beta}$. If $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ is a solitary wave solution to (2.2), then $\boldsymbol{\phi}_\omega$ satisfies $\mathbf{J}\mathbf{L}_-(\omega)\boldsymbol{\phi}_\omega=0$.

The linearization at the solitary wave (2.4) takes the form

$$\partial_t\boldsymbol{\rho}=\mathbf{J}\mathbf{L}(\omega)\boldsymbol{\rho},\qquad \boldsymbol{\rho}(x,t)=\begin{bmatrix}\mathrm{Re}\,\rho(x,t)\\ \mathrm{Im}\,\rho(x,t)\end{bmatrix}\in\mathbb{R}^{2N}. \tag{4.4}$$

Both $\mathbf{L}_-$ and $\mathbf{L}$ act on $H^1(\mathbb{R}^n,\mathbb{R}^{2N})$; by $\mathbb{C}$-linearity, we extend them onto $\mathbf{X}=H^1(\mathbb{R}^n,\mathbb{C}^{2N})=H^1(\mathbb{R}^n,\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^{2N})$.

Theorem 4.1 (Weyl's essential spectrum theorem, [RS78], Theorem XIII.14, Corollary 2)**.** *Let* A *be a self-adjoint operator and let* C *be a relatively compact perturbation of* A*. Then:*

- $B=A+C$ *defined with* $D(B)=D(A)$ *is a closed operator.*
- $\sigma_{\mathrm{ess}}(B)=\sigma_{\mathrm{ess}}(A)$.

Applying Theorem 4.1 to $A=\mathrm{i}\mathbf{J}(\mathbf{D}_m-\omega)$, $B=\mathrm{i}(\mathbf{J}\mathbf{L}(\omega)-A)$, we conclude:

Lemma 4.1. $\sigma_{\mathrm{ess}}(\mathbf{J}\mathbf{L}(\omega))=\mathrm{i}(\mathbb{R}\backslash(-m+|\omega|,m-|\omega|))$.

4.2 Independence on the choice of Dirac matrices

Let $P \in \mathrm{End}(\mathbb{C}^N)$ be a hermitian projector which commutes with all Dirac matrices: $P^2 = P$, $P^* = P$, $[P, \alpha^\jmath] = 0$, $\quad 1 \le \jmath \le n$, $[P, \beta] = 0$. Denote $\mathscr{X}_0 = P\mathscr{X}$, $\mathscr{X}_1 = (1-P)\mathscr{X}$. Denote $\mathbf{P} = \begin{bmatrix} \mathrm{Re}\, P & -\mathrm{Im}\, P \\ \mathrm{Im}\, P & \mathrm{Re}\, P \end{bmatrix}$. $\mathbf{P}$ commutes with $\boldsymbol{\alpha}^\jmath$ and with $\boldsymbol{\beta}$. Since P is $\mathbb{C}$-linear, $\mathbf{P}$ commutes with $\mathbf{J}$. The relation $P\phi = \phi$ leads to $\mathbf{P}\boldsymbol{\phi} = \boldsymbol{\phi}$.

Lemma 4.2. *If $P\phi_\omega = \phi_\omega$, then the linearization at $\phi_\omega e^{-\mathrm{i}\omega t}$ satisfies:*

1. $\sigma_{\mathrm{ess}}(\mathbf{JL}) = \sigma_{\mathrm{ess}}(\mathbf{JL}_-) = \sigma_{\mathrm{ess}}(\mathbf{JL}|_{\mathbf{X}_0}) = \sigma_{\mathrm{ess}}(\mathbf{JL}|_{\mathbf{X}_1}) = \mathrm{i}\big(\mathbb{R} \setminus (-m + |\omega|, m - |\omega|)\big)$*;*
2. $\sigma(\mathbf{JL}|_{\mathbf{X}_1}) = \sigma(\mathbf{JL}_-) \subset \mathrm{i}\mathbb{R}$*;*
3. $\sigma_p(\mathbf{JL}|_{\mathbf{X}}) = \sigma_p(\mathbf{JL}_-) \cup \sigma_p(\mathbf{JL}|_{\mathbf{X}_0})$, $\quad \sigma_p(\mathbf{JL}|_{\mathbf{X}}) \setminus \mathrm{i}\mathbb{R} = \sigma_p(\mathbf{JL}|_{\mathbf{X}_0})$.

Proof. The statement about the essential spectrum is immediate due to Lemma 4.1. The inclusion $\sigma(\mathbf{JL}_-) \subset \mathrm{i}\mathbb{R}$ follows from $\mathbf{L}_-$ being self-adjoint (with its spectrum a subset of $\mathbb{R}$) and commuting with $\mathbf{J}$ (which has $\pm\mathrm{i}$ as its eigenvalues when acting on $\mathbb{C}^{2N} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^{2N}$).

$\mathbf{JL}$ acts invariantly in both $\mathbf{X}_0$ and $\mathbf{X}_1$. Due to this invariance, one has $\sigma(\mathbf{JL}) = \sigma(\mathbf{JL}|_{\mathbf{X}_0}) \cup \sigma(\mathbf{JL}|_{\mathbf{X}_1})$. Moreover, $\mathbf{JL}|_{\mathbf{X}_1} = \mathbf{JL}_-|_{\mathbf{X}_1}$, hence $\sigma(\mathbf{JL}|_{\mathbf{X}_1}) = \sigma(\mathbf{JL}_-|_{\mathbf{X}_1}) \subset \mathrm{i}\mathbb{R}$. We conclude that $\sigma(\mathbf{JL}) \setminus \mathrm{i}\mathbb{R} = \sigma(\mathbf{JL}|_{\mathbf{X}_0}) \setminus \mathrm{i}\mathbb{R}$. ☐

Remark 4.1. Since $\sigma(\mathbf{JL}_-) \subset \mathrm{i}\mathbb{R}$, it follows that $\sigma(\mathbf{JL}|_{\mathrm{Range}\ \mathbf{P}}) \setminus \mathrm{i}\mathbb{R} = \sigma(\mathbf{JL}) \setminus \mathrm{i}\mathbb{R}$. Therefore, the linearization of (2.2) at the solitary wave $\phi_\omega(x)e^{-\mathrm{i}\omega t}$ has the same point spectrum away from the imaginary axis as the linearization at its embedding into the space of spinors of higher dimension. In particular, if there were a family of eigenvalues λ_j of $\mathbf{JL}(\omega_j)|_{\mathrm{Range}\ \mathbf{P}}$ bifurcating from $\lambda_b \in \sigma_{\mathrm{ess}}(\mathbf{JL}(\omega_b)|_{\mathrm{Range}\ \mathbf{P}})$ such that $\lambda_b \notin \sigma_p(\mathbf{JL}|_{\mathrm{Range}\ \mathbf{P}})$, then there would be the same family of eigenvalues of $\mathbf{JL}(\omega)$ bifurcating from $\lambda_b \in \sigma_{\mathrm{ess}}(\mathbf{JL}(\omega_b))$, $\lambda_b \notin \sigma_p(\mathbf{JL}(\omega_b))$. Therefore, before studying the bifurcations of point eigenvalues, we can first embed the Dirac equation and a particular solitary wave solution into the spinor space of higher dimension.

Let us show that there is no dependence on which embedding we choose, as long as n is odd.

Lemma 4.3 (Dirac-Pauli theorem)**.** *Let $\{\alpha^\jmath,\ 1 \le \jmath \le n;\ \beta\}$ and $\{\tilde\alpha^\jmath,\ 1 \le \jmath \le n;\ \tilde\beta\}$, be two sets of the Dirac matrices of the same dimension N:*

$$\{\alpha^\jmath, \alpha^k\} = 2\delta_{\jmath k}, \quad \{\alpha^\jmath, \beta\} = 0; \qquad \{\tilde\alpha^\jmath, \tilde\alpha^k\} = 2\delta_{\jmath k}, \quad \{\tilde\alpha^\jmath, \tilde\beta\} = 0.$$

1. *Let $n = 2d + 1$, $d \in \mathbb{N}$. There is an invertible matrix S such that*
$$\tilde\alpha^\jmath = S^{-1}\alpha^\jmath S, \qquad 1 \le \jmath \le n; \qquad \tilde\beta = S^{-1}\beta S.$$
2. *Let $n = 2d$, $d \in \mathbb{N}$. There is an invertible matrix S and $\sigma \in \{\pm 1\}$ such that*
$$\tilde\alpha^\jmath = \sigma S^{-1}\alpha^\jmath S, \qquad 1 \le \jmath \le n; \qquad \tilde\beta = \sigma S^{-1}\beta S.$$

Above, one could choose S to be unitary. See [Pau36, vdW32, Dir28], [Tha92, Lemma 2.25], and also [Kes61, Theorem 7] for general version in odd spatial dimensions.

Let us give the sketch of the construction from [Fed96].

Proof. Let us remind the standard construction of the irreducible representation of the Clifford algebra with $2d$ generators. Let $\boldsymbol{e}_j$ and $\boldsymbol{f}_j$, $1 \le j \le d$, be the generators of the Clifford algebra Cl_{2d}:

$$\{\boldsymbol{e}_j, \boldsymbol{e}_k\} = \{\boldsymbol{f}_j, \boldsymbol{f}_k\} = 2\delta_{jk}, \qquad \{\boldsymbol{e}_j, \boldsymbol{f}_k\} = 0; \qquad 1 \le j,\, k \le d.$$

Define $\boldsymbol{z}_j = \frac{1}{2}(\boldsymbol{e}_j + \mathrm{i}\boldsymbol{f}_j)$, $z_j^* = \frac{1}{2}(-\boldsymbol{e}_j + \mathrm{i}\boldsymbol{f}_j)$. Then $\boldsymbol{z}_j^2 = (\boldsymbol{z}_j^*)^2 = 0$, $\{\boldsymbol{z}_j, \boldsymbol{z}_k^*\} = \delta_{jk}$. The operators $\boldsymbol{z}_j$ and $\boldsymbol{z}_j^*$ are often referred to as the operators of creation and annihilation. Define the "vacuum vector" $\boldsymbol{p}_0 = \prod_{j=1}^d (\boldsymbol{z}_j \boldsymbol{z}_j^*) \in Cl_{2d}$, and let $\mathbf{S} = \{\boldsymbol{v}\boldsymbol{p}_0\ ;\ \boldsymbol{v} \in Cl_{2d}\} \subset Cl_{2d}$ be the left ideal of $\boldsymbol{p}_0$. Since $\boldsymbol{z}_j\boldsymbol{p}_0 = 0$ for $1 \le j \le d$, the elements of the form $(\boldsymbol{z}_1^*)^{a_1} \dots (\boldsymbol{z}_d^*)^{a_d}\boldsymbol{p}_0$, with $a_1, \dots, a_d \in \{0, 1\}$, form the basis of the spinor space $\mathbf{S}$; hence, $\dim \mathbf{S} = 2^d$. The action of Cl_{2d} in $\mathbf{S}$ is the only irreducible representation of Cl_{2d}. Any other representation of Cl_{2d} is isomorphic to this representation or to its several copies.

In the case of the even-dimensional space, $x \in \mathbb{R}^n$ with $n = 2d$, we need consider the Clifford algebra Cl_{2d+1}. We will denote its generators by $\boldsymbol{e}_j$, $1 \le j \le d+1$, and $\boldsymbol{f}_k$, $1 \le k \le d$:

$$\{\boldsymbol{e}_j, \boldsymbol{e}_{j'}\} = 2\delta_{jj'}, \qquad \{\boldsymbol{f}_k, \boldsymbol{f}_{k'}\} = 2\delta_{kk'}, \qquad \{\boldsymbol{e}_j, \boldsymbol{f}_k\} = 0; \qquad 1 \le j,\, j' \le d+1, \quad 1 \le k,\, k' \le d.$$

We consider Cl_{2d+1} as embedded into Cl_{2d+2}, adding one more generator $\boldsymbol{f}_{d+1}$ which anticommutes with $\boldsymbol{e}_j$, $1 \le j \le d+1$ and with $\boldsymbol{f}_j$, $1 \le j \le n$, and also satisfies $\boldsymbol{f}_{d+1}^2 = 1$. We introduce $\boldsymbol{z}_j = \frac{1}{2}(\boldsymbol{e}_j + \mathrm{i}\boldsymbol{f}_j)$ and $\boldsymbol{z}_j^* = \frac{1}{2}(-\boldsymbol{e}_j + \mathrm{i}\boldsymbol{f}_j)$, $1 \le j \le d+1$, and build the representation space $\mathbf{S}$ which is the left ideal of the element $\prod_{j=1}^{d+1} \boldsymbol{z}_j(\boldsymbol{z}_j)^*$, $\dim \mathbf{S} = 2^{d+1}$. The action of Cl_{2d+1} in $\mathbf{S}$ is reducible. We decompose $\mathbf{S}$ into $\mathbf{S} = \mathbf{S}_0 \otimes \mathbf{S}_1$, where $\mathbf{S}_0$ consists of elements which can be represented as the products of a certain number of $\boldsymbol{e}_j$, $1 \le j \le d+1$, and $\boldsymbol{f}_j$, $1 \le j \le d$, while $\mathbf{S}_1$ consists of elements which can be represented as the products of some $\boldsymbol{e}_j$, $1 \le j \le d+1$, some $\boldsymbol{f}_j$, $1 \le j \le d$, and one copy of the element $\boldsymbol{f}_{d+1}$. Then Cl_{2d+1} acts invariantly in $\mathbf{S}_0$ and $\mathbf{S}_1$. Moreover, each of these representations is irreducible since $\dim \mathbf{S}_0 = \dim \mathbf{S}_1 = 2^d$, while the only irreducible representation of the subalgebra $Cl_{2d} \subset Cl_{2d+1}$ is of dimension 2^d. The representations of Cl_{2d+1} in $\mathbf{S}_0$ and $\mathbf{S}_1$ are not isomorphic. □

Remark 4.2. The representations of Cl_{2d+1} in $\mathbf{S}_0$ and $\mathbf{S}_1$ are related as follows. The left multiplication by $\boldsymbol{f}_{d+1}$ defines the mapping

$$\mu : \mathbf{S}_0 \to \mathbf{S}_1, \qquad \mu : \mathfrak{g} \mapsto \boldsymbol{f}_{d+1}\mathfrak{g} \in \mathbf{S}_1,$$

with the inverse

$$\mu^{-1} : \mathbf{S}_1 \to \mathbf{S}_0, \qquad \mu^{-1} : \mathfrak{h} \mapsto \boldsymbol{f}_{d+1}\mathfrak{h} \in \mathbf{S}_0.$$

Due to the relations

$$\boldsymbol{f}_{d+1}\boldsymbol{e}_j\boldsymbol{f}_{d+1} = -\boldsymbol{e}_j, \quad 1 \le j \le d+1, \qquad \boldsymbol{f}_{d+1}\boldsymbol{f}_j\boldsymbol{f}_{d+1} = -\boldsymbol{f}_j, \quad 1 \le j \le d,$$

the action of $\boldsymbol{e}_j$, $1 \le j \le d+1$, and $\boldsymbol{f}_j$, $1 \le j \le d$, by the left multiplication on $\mathbf{S}_0$ and on $\mathbf{S}_1$ are related by

$$\mu \circ L_{\mathbf{S}_0}(\boldsymbol{e}_j) \circ \mu^{-1} = -L_{\mathbf{S}_1}(\boldsymbol{e}_j), \quad 1 \le j \le d+1, \qquad \mu \circ L_{\mathbf{S}_0}(\boldsymbol{f}_j) \circ \mu^{-1} = -L_{\mathbf{S}_1}(\boldsymbol{f}_j), \quad 1 \le j \le d.$$

Lemma 4.4. *The spectrum of* $\mathbf{JL}$ *is symmetric with respect to the real and imaginary axes.*

Proof. Due to Lemma 4.1, it is enough to consider the discrete spectrum. The inclusion $-\bar\lambda \in \sigma(\mathbf{JL})$ follows from $(-\mathbf{JL})^*$ being the conjugate to $\mathbf{JL}$:

$$(-\mathbf{JL})^* = \mathbf{L}^*(-\mathbf{J})^* = \mathbf{LJ} = \mathbf{J}^{-1}(\mathbf{JL})\mathbf{J}.$$

Let Γ be a self-adjoint matrix which satisfies $\Gamma^2 = 1$, anticommutes with $\alpha^\jmath$, β, and also satisfies $\Gamma\phi = \phi$. (Such a matrix exists if we embed the Dirac equation into the spinor space of sufficiently large dimension; by Lemma 4.2, this does not change the spectrum.) Then $\boldsymbol{\Gamma} = \begin{bmatrix} \operatorname{Re}\Gamma & -\operatorname{Im}\Gamma \\ \operatorname{Im}\Gamma & \operatorname{Re}\Gamma \end{bmatrix}$ is a self-adjoint matrix which satisfies $\boldsymbol{\Gamma}^2 = 1$, anticommutes with $\boldsymbol{\alpha}^\jmath$, $\boldsymbol{\beta}$, commutes with $\mathbf{J}$ (since Γ is $\mathbb{C}$-invariant), and also satisfies $\boldsymbol{\Gamma\phi} = \boldsymbol{\phi}$, where $\boldsymbol{\phi} = \begin{bmatrix} \operatorname{Re}\phi \\ \operatorname{Im}\phi \end{bmatrix}$. Then $\boldsymbol{\Gamma}\mathbf{JL}\boldsymbol{\Gamma}^{-1} = -\mathbf{JL}$, showing that $\lambda \in \sigma_p(\mathbf{JL})$ implies $-\lambda \in \sigma_p(\mathbf{JL})$. □

4.3 Point spectrum of the linearization operator

Lemma 4.5. *Let* α^0 *be an hermitian matrix anticommuting with* $\alpha^\jmath$, $1 \le \jmath \le n$, *and with* β. *Then* $\alpha^0\phi$ *is an eigenfunction of* $\mathcal{L}_-$ *and of* $\mathcal{L}$, *corresponding to the eigenvalue* $\lambda = -2\omega$.

Remark 4.3. If $n = 3$, one can take $\alpha^0 = \alpha^1\alpha^2\alpha^3\beta$; with the standard choice of Dirac matrices, $\alpha^0 = \mathrm{i}\begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}$.

Proof. Since α^0 anticommutes with $\alpha^\jmath$, $1 \le \jmath \le n$, and with β, and taking into account (3.1), we have:

$$\mathcal{L}_-\alpha^0\phi = (D_m - \omega - f(\phi^*\beta\phi)\beta)\alpha^0\phi = \alpha^0(-D_m - \omega + f(\phi^*\beta\phi)\beta)\phi = \alpha^0(-\mathcal{L}_- - 2\omega)\phi = -2\omega\alpha^0\phi.$$

Since α^0 and β are Hermitian, $2\operatorname{Re}[\phi^*\beta\alpha^0\phi] = \phi^*\beta\alpha^0\phi + \overline{\phi^*\beta\alpha^0\phi} = \phi^*\{\beta, \alpha^0\}\phi = 0$; therefore, one also has $\mathcal{L}\alpha^0\phi = \mathcal{L}_-\alpha^0\phi = -2\omega\alpha^0\phi$. □

It follows that the linearization operator has an eigenvalue $2\omega\mathrm{i}$:

$$2\omega\mathrm{i} \in \sigma_p(\mathcal{J}\mathcal{L}(\omega)).$$

Since $\sigma(\mathbf{J}\mathbf{L}(\omega))$ is symmetric with respect to $\mathbb{R}$ and $\mathrm{i}\mathbb{R}$, for any $f(\eta)$ in (2.2) and in any dimension $n \geq 1$, we arrive at the following result:

Lemma 4.6. $\pm 2\omega\mathrm{i} \in \sigma_p(\mathbf{J}\mathbf{L}(\omega))$.

Remark 4.4. For $|\omega| > m/3$, the eigenvalues $\pm 2\omega\mathrm{i}$ are embedded into the essential spectrum.

Remark 4.5. The result of Lemma 4.6 takes place for any nonlinearity $f(\psi^*\beta\psi)$ and in any dimension. The spatial dimension n and the number of components of ψ could be such that there is no matrix α^0 which anticommutes with $\alpha^\jmath$, $1 \leq \jmath \leq n$, and with β; then the eigenvector corresponding to $\pm 2\omega\mathrm{i}$ can be constructed either using the spatial reflections. Alternatively one can double the size of the spinors so that there is a desired matrix α^0; by the results of this section, this does not change the spectrum of the linearized operator.

5 Properties of embedded eigenstates

5.1 Decay of embedded eigenstates before the embedded threshold

Lemma 5.1. *Let $\lambda \in \sigma_p(\mathbf{J}\mathbf{L}(\omega))$ satisfy $\lambda \in \mathrm{i}\mathbb{R}$, $m - |\omega| \leq |\lambda| < m + |\omega|$. Then the corresponding eigenfunction is exponentially decaying.*

Remark 5.1. The proof for the case $n = 1$ follows from the properties of the Jost solutions (any eigenfunction can be decomposed at $x = \pm\infty$ over the exponentially decaying Jost solutions, and only the ones exponentially decreasing at $\pm\infty$ can participate; thus, the exponential decay of Jost solutions translates into the exponential decay of eigenfunctions).

Proof. The eigenfunction $\boldsymbol{\zeta} \in L^2(\mathbb{R}^n, \mathbb{C}^{2N})$, $\|\boldsymbol{\zeta}\| = 1$, which corresponds to λ, satisfies $(\mathbf{D}_m - \omega + \lambda\mathbf{J})\boldsymbol{\zeta} = -\mathbf{V}\boldsymbol{\zeta}$. Applying $\Pi^\pm$, we get:

$$(\mathbf{D}_m - \omega + \mathrm{i}\lambda)\boldsymbol{\zeta}^+ = -\Pi^+\mathbf{V}\boldsymbol{\zeta}, \qquad (\mathbf{D}_m - \omega - \mathrm{i}\lambda)\boldsymbol{\zeta}^- = -\Pi^-\mathbf{V}\boldsymbol{\zeta}.$$

Assume that $\mathrm{i}\lambda$ is of the same sign as ω (the other case is treated verbatim by exchanging the treatment of $\boldsymbol{\zeta}^\pm$); then $\omega - \mathrm{i}\lambda$ is in the spectral gap of $\mathbf{D}_m$, hence we can write

$$\boldsymbol{\zeta}^+ = -(\mathbf{D}_m - \omega + \mathrm{i}\lambda)^{-1}\Pi^+\mathbf{V}\boldsymbol{\zeta}.$$

The Combes-Thomas method [His00], as in the proof of Lemma 3.1, shows that $\boldsymbol{\zeta}^+$ is exponentially decaying.

Now we turn to $\boldsymbol{\zeta}^-$. Due to the exponential decay of $\phi_\omega(x)$ (see Lemma 3.1) and hence of $\mathbf{V}(x,\omega)$, we can choose $\tau > 0$ small enough so that

$$\sup_{x\in\mathbb{R}^n,\,\omega\in\mathcal{I}} \|r^{1/2}e^{\tau|x|}\Pi^-\mathbf{V}(x,\omega)\|_{\mathrm{End}(\mathbb{C}^{2N})} < \infty.$$

Let $\Theta(x) \in C^\infty(\mathbb{R}^n)$ be such that $\Theta \equiv 1$ for $|x| \geq 2$, $\Theta \equiv 0$ for $|x| \leq 1$. Applying Lemma B.4, we have:

$$\|e^{\tau r}\Theta(x)\boldsymbol{\zeta}^-\| \leq C\|r^{1/2}e^{\tau r}(\mathbf{D}_m - \omega - \mathrm{i}\lambda)\Theta(x)\boldsymbol{\zeta}^-\| \leq C\|\Theta(x)r^{1/2}e^{\tau r}\Pi^-\mathbf{V}\boldsymbol{\zeta}\| + C\|r^{1/2}e^{\tau r}[\Theta(x), \mathbf{D}_0]\boldsymbol{\zeta}^-\|.$$

It follows that $e^{\tau|x|}\boldsymbol{\zeta}^-$ is bounded in L^2.

Taking derivatives in x and using the boundedness of $\boldsymbol{\zeta}$ in $H^s(\mathbb{R}^n, \mathbb{C}^{2N})$, $\forall s \in \mathbb{N}$, we conclude that $e^{\tau|x|}\Theta(x)\boldsymbol{\zeta}^-(x) \in H^s(\mathbb{R}^n, \mathbb{C}^{2N})$, $\forall s \in \mathbb{N}$. The exponential decay of $\boldsymbol{\zeta}^-$ and hence of $\boldsymbol{\zeta}$ follows. ∎

5.2 Absence of embedded eigenvalues beyond the embedded threshold

Lemma 5.2. *Assume that there are $\epsilon > 0$ and $C < \infty$ such that $|\mathbf{V}(x)| \le Ce^{-\epsilon|x|}$, $x \in \mathbb{R}^n$. If $\lambda \in \sigma_p(\mathbf{JL}) \cap \mathrm{i}\mathbb{R}$ and $|\lambda| > m + |\omega|$, then for any $N > 0$ there is $C_N > 0$ such that the corresponding eigenfunction $\boldsymbol{\zeta}$ satisfies*

$$|\boldsymbol{\zeta}(x)| \le C_N e^{-N|x|}, \qquad x \in \mathbb{R}^n.$$

Remark 5.2. The proof for the case $n = 1$ follows from the properties of the Jost solutions: while any eigenfunction can be decomposed at $x = \pm\infty$ over the exponentially decaying Jost solutions, there are no decaying Jost solutions at thresholds $\lambda = \pm\mathrm{i}(m + |\omega|)$.

Proof. The proof is a bootstrap argument based on Lemma B.4. As both $\omega \pm \operatorname{Im}\lambda$ have modulus bigger than m, from this lemma we deduce that if $e^{\tau r}\big(J(D_m - \omega) - \lambda\big)\eta\boldsymbol{\zeta}$ is square integrable for any bounded smooth function η with support outside some sufficiently large ball then $e^{\tau r}\eta\boldsymbol{\zeta}$ is as well. As

$$\big(\mathbf{J}(\mathbf{D}_m - \omega) - \lambda\big)(\eta\boldsymbol{\zeta}) = \mathbf{J}\mathrm{i}\alpha \cdot (\nabla\eta)\boldsymbol{\zeta} - \mathbf{J}\mathbf{V}(\omega)\eta\boldsymbol{\zeta},$$

the bootstrap starts with the exponential decay of V. ◻

Lemma 5.3. *Let $n \ge 1$. Fix $\omega \in \mathcal{O}$. There are $C < \infty$ and $R_0 > 0$, dependent on n, ω, and λ, such that for any $R \ge R_0$ and any $u \in H_0^1(\Omega_R, \mathbb{C}^N)$,*

$$\tau^{1/2}\|e^{\tau r}u\| \le C\|r^{1/2}e^{\tau r}(\mathcal{L}(\omega) \pm \mathrm{i}\lambda)u\|, \qquad \tau \ge 1,$$

where $\mathcal{L}(\omega)u = D_m u - \omega u - f(\phi_\omega^*\beta\phi_\omega)u - 2f'(\phi_\omega^*\beta\phi_\omega)\operatorname{Re}(\phi_\omega^*\beta u)\phi_\omega$.

Proof. This is the adaptation of the Carleman-Berthier-Georgescu estimates. By Lemma B.4, for $\tau \ge 2m$ and any $u \in H_0^1(\Omega_R, \mathbb{C}^N)$, one has:

$$\begin{aligned}
&\|\big((\omega \mp \mathrm{i}\lambda)^2 - m^2 + \frac{\tau^2}{2}\big)^{1/2}e^{\tau r}u\| \le \|\mu(r)e^{\tau r}(D_m - \omega \pm \mathrm{i}\lambda)u\| \\
&\le \|\mu(r)e^{\tau r}(\mathcal{L}(\omega) \pm \mathrm{i}\lambda)u\| + \|\mu(r)e^{\tau r}(f(\phi^*\beta\phi)u + 2f'(\phi^*\beta\phi)\operatorname{Re}(\phi^*\beta u)\phi)\|,
\end{aligned}$$

with

$$\mu(r) = \Big(\frac{n}{4} + 2\frac{(\omega \mp \mathrm{i}\lambda)^2 r}{\tau} + 2r\tau\Big)^{1/2}.$$

Note that $(\omega \mp \mathrm{i}\lambda)^2 - m^2 > 0$ since $\lambda \in \mathrm{i}\mathbb{R}$, $|\lambda| > m + |\omega|$. Therefore, once ω and λ are fixed, if $R > 0$ is sufficiently large, there is $C < \infty$ so that

$$\|e^{\tau r}u\| \le C\|\mu(r)e^{\tau r}(\mathcal{L}(\omega) \mp \mathrm{i}\lambda)u\|, \qquad u \in H_0^1(\Omega_R, \mathbb{C}^N), \quad \tau \ge 1.$$

◻

Lemma 5.4. *Let $n \ge 1$. The operator $\mathbf{JL}(\omega)$ has no embedded eigenvalues $\lambda \in \mathrm{i}\mathbb{R}$ with $|\lambda| > m + |\omega|$.*

Remark 5.3. For the case $n = 1$, the proof follows from the analysis of the Jost solutions, and the conclusion is stronger: the operator $\mathbf{JL}(\omega)$ has no embedded eigenvalues $\lambda \in \mathrm{i}\mathbb{R}$ with $|\lambda| \ge m + |\omega|$.

Proof. Assume that $\lambda \in \mathrm{i}\mathbb{R}$, $|\lambda| > m + |\omega|$, is an embedded eigenvalue of $\mathbf{JL}(\omega)$, with $\boldsymbol{\zeta} \in L^2(\mathbb{R}^n, \mathbb{C}^{2N})$ the corresponding eigenvector:

$$-\mathbf{J}\lambda\boldsymbol{\zeta} = \mathbf{D}_m\boldsymbol{\zeta} - \omega\boldsymbol{\zeta} - f(\boldsymbol{\phi}^*\boldsymbol{\beta}\boldsymbol{\phi})\boldsymbol{\beta}\boldsymbol{\zeta} - 2f'(\boldsymbol{\phi}^*\boldsymbol{\beta}\boldsymbol{\phi})\operatorname{Re}(\boldsymbol{\phi}^*\boldsymbol{\beta}\boldsymbol{\zeta})\boldsymbol{\beta}\boldsymbol{\phi}. \tag{5.1}$$

Let $\Theta \in C^\infty(\mathbb{R}^n)$ be a smooth radially symmetric cut-off function with support in the closure of Ω_{R+1} and with value 1 in Ω_{R+2}. By Lemma 5.3,

$$\forall\tau \ge 1, \qquad \|e^{\tau r}\Theta\boldsymbol{\zeta}\| \le \frac{C}{\sqrt{\tau}}\|r^{1/2}e^{\tau r}(\mathbf{JL} - \lambda)\Theta\boldsymbol{\zeta}\|. \tag{5.2}$$

Since $\mathbf{JL}\boldsymbol{\zeta} = \lambda\boldsymbol{\zeta}$, we have $(\mathbf{JL} - \lambda)\Theta\boldsymbol{\zeta} = [\mathbf{JL}, \Theta]\boldsymbol{\zeta}$. By (5.2),

$$\forall\tau \geq 1, \quad \forall k > R, \quad \|e^{\tau r}\Theta\boldsymbol{\zeta}\| \leq \frac{C}{\sqrt{\tau}}\|r^{1/2}e^{\tau r}[\mathbf{JL}, \Theta]\boldsymbol{\zeta}\|.$$

Taking into account that $\partial_{x_j}\Theta$ and hence $[\mathbf{JD}, \Theta]$ are zero outside of the ball B_{R+2}, we conclude that

$$\forall\tau \geq 1, \quad \forall k > R, \quad \|e^{\tau r}\boldsymbol{\zeta}\|_{L^2(\Omega_{R+2},\mathbb{C}^{2N})} \leq \frac{C}{\sqrt{\tau}}e^{(R+2)\tau}\|r^{1/2}\nabla\Theta\|_{L^\infty}\|\boldsymbol{\zeta}\|_{L^2(B_{R+2},\mathbb{C}^{2N})}.$$

Since $\tau > 1$ could be arbitrarily large, we conclude that $\operatorname{supp}\boldsymbol{\zeta} \cap \Omega_{R+2} = \emptyset$. The unique continuation principle, Lemma 3.4, ensures that $\boldsymbol{\zeta} \equiv 0$, contradicting our assumption that there were an embedded eigenvalue $\lambda \in \mathrm{i}\mathbb{R}$, $|\lambda| > m + |\omega|$. □

Theorem 2.2 follows from Lemmas 5.1 and 5.4.

6 Bifurcations of eigenvalues from the essential spectrum

Lemma 6.1. *Let* $\mathbf{J} \in \operatorname{End}(\mathbb{C}^{2N})$ *be skew-adjoint and invertible and let* $\mathbf{L}$ *be self-adjoint on* $L^2(\mathbb{R}^n, \mathbb{C}^{2N})$*. If* $\lambda \in \sigma_p(\mathbf{JL}) \setminus \mathrm{i}\mathbb{R}$ *with the corresponding eigenvector* $\boldsymbol{\zeta}$*, then*

$$\langle\boldsymbol{\zeta}, \mathbf{L}\boldsymbol{\zeta}\rangle = 0, \qquad \langle\boldsymbol{\zeta}, \mathbf{J}\boldsymbol{\zeta}\rangle = 0.$$

Proof. One has $\mathbf{JL}\boldsymbol{\zeta} = \lambda\boldsymbol{\zeta}$, $\mathbf{L}\boldsymbol{\zeta} = \lambda\mathbf{J}^{-1}\boldsymbol{\zeta}$, hence

$$\langle\boldsymbol{\zeta}, \mathbf{L}\boldsymbol{\zeta}\rangle = \lambda\langle\boldsymbol{\zeta}, \mathbf{J}^{-1}\boldsymbol{\zeta}\rangle. \tag{6.1}$$

Since $\langle\boldsymbol{\zeta}, \mathbf{L}\boldsymbol{\zeta}\rangle \in \mathbb{R}$ and $\langle\boldsymbol{\zeta}, \mathbf{J}^{-1}\boldsymbol{\zeta}\rangle \in \mathrm{i}\mathbb{R}$, the condition $\operatorname{Re}\lambda \neq 0$ implies that both sides in (6.1) are equal to zero. □

Remark 6.1. If an eigenvector $\boldsymbol{\zeta}$ corresponding to $\lambda \in \sigma_p(\mathbf{JL})$ satisfies $\langle\boldsymbol{\zeta}, \mathbf{L}\boldsymbol{\zeta}\rangle = 0$, we will say that λ has zero Krein signature. The Krein signature is only interesting for $\lambda \in \mathrm{i}\mathbb{R}$ since, according to Lemma 6.1, all eigenvalues of $\mathbf{JL}$ with nonzero real part have zero Krein signature.

We will formulate our results for the operator $\mathbf{L}(\omega) = \mathbf{D}_m - \omega + \mathbf{V}(\omega)$, $\omega \in [-m, m]$, with $\mathbf{V}$ operator-valued, with $\mathbf{V}(\omega)$ zero order and self-adjoint. for each $\omega \in [-m, m]$. Note that $\mathbf{L}(\omega)$ is not necessarily a linearization at a solitary wave of the nonlinear Dirac equation.

Lemma 6.2. *Let* $n \geq 1$*. Let* $\mathbf{J} \in \operatorname{End}(\mathbb{C}^{2N})$ *be skew-adjoint and invertible and let*

$$\mathbf{L}(\omega) = \mathbf{D}_m - \omega + \mathbf{V}(\omega), \qquad \omega \in [-m, m],$$

with $\mathbf{V} : [-m, m] \to L^\infty(\mathbb{R}^n, \operatorname{End}(\mathbb{C}^{2N}))$ *a zero-order operator-valued function which is self-adjoint for each* $\omega \in [-m, m]$*. Let* $\omega_b \in [-m, m]$*, and assume that there is* $\varepsilon > 0$ *such that*

$$\limsup_{\omega\to\omega_b} \|\langle Q\rangle^{1+\varepsilon}\mathbf{V}(\omega)\|_{L^\infty(\mathbb{R}^n,\operatorname{End}(\mathbb{C}^{2N}))} < \infty. \tag{6.2}$$

Let $\omega_j \in (-m, m)$*,* $\omega_j \underset{j\to\infty}{\longrightarrow} \omega_b$*. Let* $\lambda_j \in \sigma_d(\mathbf{JL}(\omega_j))$ *be a family of eigenvalues such that* $\operatorname{Re}\lambda_j \neq 0$*,* $\lambda_j \underset{j\to\infty}{\longrightarrow} \lambda_b \in \mathrm{i}\mathbb{R}$*. If* $\omega_b \in \{0; \pm m\}$*, additionally assume that*

$$\lambda_b \neq \pm\mathrm{i}(m - |\omega_b|). \tag{6.3}$$

Then

$$\lambda_b \in \sigma_p(\mathbf{JL}(\omega_b)).$$

Moreover, there is an infinite subsequence of eigenfunctions $(\boldsymbol{\zeta}_j)_{j\in\mathbb{N}}$ *corresponding to* $\lambda_j \in \sigma_p(\mathbf{JL}(\omega_j))$ *which converges to the eigenfunction corresponding to* λ_b*, and this eigenfunction* $\boldsymbol{\zeta}_b$ *satisfies*

$$\langle\boldsymbol{\zeta}_b, \mathbf{L}(\omega_b)\boldsymbol{\zeta}_b\rangle = 0, \qquad \langle\boldsymbol{\zeta}_b, \mathbf{J}\boldsymbol{\zeta}_b\rangle = 0.$$

Remark 6.2. In this lemma, we do not include bifurcations from thresholds in full generality, but if we replace (6.2) by a stronger condition that there is $\varepsilon > 0$ such that

$$\limsup_{\omega\to\omega_b} \|\langle Q\rangle^{2+\varepsilon}\mathbf{V}(\omega)\|_{L^\infty(\mathbb{R}^n,\mathrm{End}(\mathbb{C}^{2N}))} < \infty, \tag{6.4}$$

then the lemma remains true without any restriction on ω_b or λ_b.

Remark 6.3. The conclusion of the lemma is trivial for $\lambda_b \in \mathrm{i}\mathbb{R}$ with $|\lambda_b| < m - |\omega_b|$, when λ_b is not in the essential spectrum, and the inclusion $\lambda_b \in \sigma_p(\mathbf{JL}(\omega_b))$ follows from the continuous dependence of eigenvalues in the discrete spectrum on a parameter, ω.

Remark 6.4. Due to the exponential decay of ϕ_ω proved in Lemma 3.1, the assumption (6.2) is trivially satisfied for linearizations at solitary waves with $\omega_b \notin \partial\mathcal{O}$. Due to (3.6), the assumption (6.2) is also satisfied for the linearizations at solitary waves of the nonlinear Dirac equation from Lemma 3.5 for $\omega_b = m$. On the other hand, according to asymptotics (3.6), one sees that the condition (6.4) is not satisfied for the linearizations as $\omega \to m$. The case $\omega_b = m$ is studied in detail in Sections 7 and 8.

Remark 6.5. By Lemma 6.2, if $\lambda_b = \pm\mathrm{i}(m + |\omega_b|)$ and $\omega_b \notin \{0; \pm m\}$, then one must have $\lambda_b \in \sigma_p(\mathbf{JL}(\omega_b))$. This excludes bifurcations from $\pm\mathrm{i}(m+|\omega|)$, $|\omega| < m$, in the case $n = 1$, since, as the analysis of the Jost solutions shows, there can be no embedded eigenvalues at $\pm\mathrm{i}(m + |\omega|)$.

Proof. Let $(\boldsymbol{\zeta}_j)_{j\in\mathbb{N}}$ be a sequence of unit eigenvectors associated with eigenvalues λ_j, so that $\mathbf{JL}(\omega_j)\boldsymbol{\zeta}_j = \lambda_j\boldsymbol{\zeta}_j$. It follows that

$$(\mathbf{D}_m - \omega_j - \lambda_j\mathbf{J})\boldsymbol{\zeta}_j = -\mathbf{V}(\omega_j)\boldsymbol{\zeta}_j.$$

Let $\Pi^\pm = \frac{1}{2}(1 \mp \mathrm{i}\mathbf{J})$ be the projectors onto eigenspaces of $\mathbf{J}$ corresponding to $\pm\mathrm{i}$. We denote $\boldsymbol{\zeta}_j^\pm = \Pi^\pm\boldsymbol{\zeta}_j$. By Lemma 6.1, $0 = \langle\boldsymbol{\zeta}_j, \mathbf{J}\boldsymbol{\zeta}_j\rangle = \mathrm{i}\|\boldsymbol{\zeta}_j^+\|^2 - \mathrm{i}\|\boldsymbol{\zeta}_j^-\|^2$, $j \in \mathbb{N}$, while $1 = \|\boldsymbol{\zeta}_j\|^2 = \|\boldsymbol{\zeta}_j^+\|^2 + \|\boldsymbol{\zeta}_j^-\|^2$, $j \in \mathbb{N}$; we conclude that $\|\boldsymbol{\zeta}_j^\pm\| = 1/2$ and

$$\big((\mathbf{D}_m - \omega_j) \mp \mathrm{i}\lambda_j\big)\boldsymbol{\zeta}_j^\pm = -\Pi^\pm\mathbf{V}(\omega_j)\boldsymbol{\zeta}_j.$$

If the condition (6.3) is satisfied, then either $\mathrm{i}\lambda_b - \omega_b$ or $-\mathrm{i}\lambda_b - \omega_b$ is not a threshold of $\mathbf{D}_m$. Let us assume that $\mathrm{i}\lambda_b + \omega_b$ is not:

$$\mathrm{i}\lambda_b + \omega_b \ne \pm m.$$

Then $\mathrm{i}\lambda_j + \omega_j \ne \pm m$ if j is large enough. From Lemma A.5, we deduce that for any $s > -1/2$

$$\|\langle Q\rangle^s\boldsymbol{\zeta}_j\|_{H^1} \le c\|\langle Q\rangle^{s+1}\big((\mathbf{D}_m - \omega_j) - \mathrm{i}\lambda_j\big)\boldsymbol{\zeta}_j^+\| = C\|\langle Q\rangle^{s+1}\Pi^+\mathbf{V}(\omega_j)\boldsymbol{\zeta}_j\|,$$

for some $C < \infty$ which does not depend on j. Hence, for $s = \varepsilon$, we have

$$\|\langle Q\rangle^\varepsilon\boldsymbol{\zeta}_j^+\|_{H^1} \le cC, \qquad j \in \mathbb{N}.$$

It follows that $(\boldsymbol{\zeta}_j^+)_{j\in\mathbb{N}}$ is precompact in $L^2(\mathbb{R}^n, \mathbb{C}^{2N})$, and we can choose a subsequence which converges to a vector of norm $\lim_{j\to\infty}\|\boldsymbol{\zeta}_j^+\| = 1/2$. As $(\boldsymbol{\zeta}_j^-)$ is weakly convergent, we conclude that there exists a nonzero weak limit which is necessarily an eigenvector. □

Lemma 6.3. *Let $n \ge 1$. Let $\omega_j \in (-m, m)$, $\omega_j \xrightarrow[j\to\infty]{} \omega_b \in [-m, m]$. Let $\lambda_j \in \sigma_d(\mathbf{JL}(\omega_j))$ be a family of eigenvalues such that $\lambda_j \xrightarrow[j\to\infty]{} \lambda_b \in \mathrm{i}\mathbb{R}$. Assume that λ_b is not at any of the thresholds,*

$$|\lambda_b| \ne m \pm \omega_b,$$

and that there is $\varepsilon > 0$ such that

$$\limsup_{\omega\to\omega_b} \|\langle Q\rangle^{1+\varepsilon}\mathbf{V}(\omega)\|_{L^\infty(\mathbb{R}^n,\mathrm{End}(\mathbb{C}^{2N}))} < \infty. \tag{6.5}$$

Then $\lambda_b \in \sigma_p(\mathbf{JL}(\omega_b))$ and there is a sequence of eigenfunctions $\boldsymbol{\zeta}_j$ corresponding to $\lambda_j \in \sigma_p(\mathbf{JL}(\omega_j))$ which converges in L^2 to the eigenfunction $\boldsymbol{\zeta}_b$ corresponding to $\lambda_b \in \sigma_p(\mathbf{JL}(\omega_b))$.

If, moreover, $\operatorname{Re}\lambda_j \ne 0$, *then the corresponding eigenfunction $\boldsymbol{\zeta}_b$ satisfies $\langle\boldsymbol{\zeta}_b, \mathbf{L}(\omega_b)\boldsymbol{\zeta}_b\rangle = 0$, $\langle\boldsymbol{\zeta}_b, \mathbf{J}\boldsymbol{\zeta}_b\rangle = 0$.*

Proof. If $\mathrm{i}\lambda - \omega$ and $-\mathrm{i}\lambda - \omega$ are not thresholds of $\mathbf{D}_m$ then the proof of Lemma 6.2 shows the relative compactness of both $(\phi_j^+)_{j\in\mathbb{N}}$ and $(\phi_j^-)_{j\in\mathbb{N}}$. The rest is proved exactly as in Lemma 6.2. □

7 Bifurcations from the essential spectrum of the free Dirac equation

The limiting absorption principle for the Dirac operator has been established in [Yam73]. We need a slightly stronger version for $\lambda \in \mathcal{K}$ where $\mathcal{K}$ is a closed set which does not contain the thresholds $\pm m$, but is not necessarily compact:

Lemma 7.1. *Let $\mathcal{K} \subset \mathbb{C}$ be a closed set such that $\pm m \notin \mathcal{K}$. Then for any $s > 1/2$ there is $C < \infty$ such that*

$$\|u\|_{L^2_{-s}(\mathbb{R}^n,\mathbb{C}^N)} \le C\|(D_m - \lambda)u\|_{L^2_s(\mathbb{R}^n,\mathbb{C}^N)}, \qquad \forall \lambda \in \mathcal{K}, \qquad \forall u \in H^1_s(\mathbb{R}^n, \mathbb{C}^N). \tag{7.1}$$

Proof. For $\mathcal{K}$ compact and $\lambda \in \mathcal{K} \setminus \mathbb{R}$, this is the result of [Yam73]. For $|\lambda| \ge m+1$, $\operatorname{Im}\lambda \ne 0$, this follows from

$$(D_m - \lambda)^{-1}u = (D_m + \lambda)(-\Delta + m^2 - \lambda^2)^{-1}u,$$

where one has

$$\|(-\Delta + m^2 - \lambda^2)^{-1}\|_{L^2_s \to H^2_{-s}} = O(|\lambda^2 - m^2|^{-1/2})$$

by [Agm75, Remark 2 in Appendix A], and

$$\|D_m + \lambda\|_{H^2_{-s} \to H^1_{-s}} \le C(1 + |\lambda|).$$

This proves that for some $C < \infty$ which depends on $m \ge 0$, $s > 1/2$, and $\mathcal{K} \subset \mathbb{C}$ but not on λ, one has

$$\|(D_m - \lambda)^{-1}v\|_{H^1_{-s}} \le C\|v\|_{L^2_s}, \qquad \lambda \in \mathcal{K} \setminus \mathbb{R}, \qquad v \in L^2_s,$$

yielding (7.1) for $\operatorname{Im}\lambda \ne 0$ and for any $u \in H^1_s(\mathbb{R}^n, \mathbb{C}^N)$. By continuity, (7.1) also holds for $\lambda \in \mathcal{K}$. □

Let us consider families of eigenvalues in the limit of small amplitude solitary waves, which may be present in the spectrum up to the border of existence of solitary waves: $\omega \to \omega_b \in \{\pm m\}$. This situation could be considered as the bifurcation of eigenvalues from the continuous spectrum of the free Dirac equation.

Let $\mathbf{J} \in \operatorname{End}(\mathbb{C}^{2N})$ be skew-adjoint and invertible, and let

$$\mathbf{L}(\omega) = \mathbf{D}_m - \omega + \mathbf{V}(x, \omega), \qquad \omega \in [-m, m],$$

with $\mathbf{V} : [-m, m] \to L^\infty(\mathbb{R}^n, \operatorname{End}(\mathbb{C}^{2N}))$ an operator-valued function of $\omega \in [-m, m]$, with $\mathbf{V}(\omega)$ a zero-order operator.

Lemma 7.2. *Let $\omega_b = \pm m$. Assume that there is $s > 1/2$ such that*

$$\limsup_{\omega \to \omega_b} \|\langle Q\rangle^{2s}\mathbf{V}(\omega)\|_{L^\infty(\mathbb{R}^n,\operatorname{End}(\mathbb{C}^{2N}))} = 0. \tag{7.2}$$

Let $\omega_j \in \mathcal{O}$, $\omega_j \to \omega_b$. If $\lambda_j \in \sigma_p(\mathbf{J}\mathbf{L}(\omega_j))$, then the only possible accumulation points of $\{\lambda_j \,;\, j \in \mathbb{N}\}$ are $\lambda = \{0; \pm 2m\mathrm{i}\}$.

Remark 7.1. In this lemma, $\mathbf{V}(\omega)$ is not necessarily self-adjoint.

Remark 7.2. By (3.6), the condition (7.2) is satisfied for solitary waves from Lemma 3.5 in the nonrelativistic limit $\omega \to m$, with any $s < 1$

Proof. Let $\mathcal{K} \subset \mathbb{C}$ be a closed set such that $\pm m \notin \mathcal{K}$. According to Lemma 7.1, there is the limiting absorption principle for the free Dirac operator $D_m = -\mathrm{i}\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m$, so that the following action of its resolvent is uniformly bounded for $z \in \mathcal{K}\backslash\mathbb{R}$:

$$(D_m - z)^{-1} : \ L^2_s(\mathbb{R}^n, \mathbb{C}^N) \to L^2_{-s}(\mathbb{R}^n, \mathbb{C}^N), \qquad s > 1/2, \qquad z \in \mathcal{K}\backslash\mathbb{R}. \tag{7.3}$$

Now let $\mathcal{V} \subset \mathbb{C}$ be an arbitrary closed set which does not contain $\pm 2(m \pm \omega_b)\mathrm{i}$. To prove the theorem, we need to show that for ω sufficiently close to ω_b there is no point spectrum of $\mathbf{J}\mathbf{L}(\omega)$ in $\mathcal{V}$. Let ω be close enough to ω_b so that $\mathcal{V}$ does not contain $\pm\mathrm{i}(m \pm \omega)$. One has $\lim_{|x|\to\infty} \mathbf{L}(\omega) = \mathbf{D}_m - \omega$. Since the eigenvalues of $\mathbf{J}$ are $\pm\mathrm{i}$, the operator

$\mathbf{J}(\mathbf{D}_m-\omega)$ can be represented as the direct sum of operators $\mathrm{i}(D_m-\omega)$ and $-\mathrm{i}(D_m-\omega)$. By (7.3), the following map is bounded uniformly for $z\in\mathcal{V}\backslash\mathrm{i}\mathbb{R}$:

$$\big(\mathbf{J}(\mathbf{D}_m-\omega)-z\big)^{-1}:\ L^2_s(\mathbb{R}^n,\mathbb{C}^{2N})\to L^2_{-s}(\mathbb{R}^n,\mathbb{C}^{2N}),\qquad s>1/2,\qquad z\in\mathcal{V}\backslash\mathrm{i}\mathbb{R}. \tag{7.4}$$

The resolvent of $\mathbf{JL}(\omega)$ is expressed as

$$(\mathbf{JL}(\omega)-z)^{-1}=\big(\mathbf{J}(\mathbf{D}_m-\omega)-z\big)^{-1}\frac{1}{1+\mathbf{JV}\big(\mathbf{J}(\mathbf{D}_m-\omega)-z\big)^{-1}}. \tag{7.5}$$

Thus, the action $(\mathbf{JL}(\omega)-z)^{-1}: L^2_s(\mathbb{R}^n,\mathbb{C}^{2N})\to L^2_{-s}(\mathbb{R}^n,\mathbb{C}^{2N})$ is uniformly bounded in $z\in\mathcal{V}\backslash\mathrm{i}\mathbb{R}$ as long as the operator $\mathbf{V}(\omega):\ L^2_{-s}(\mathbb{R}^n,\mathbb{C}^{2N})\to L^2_s(\mathbb{R}^n,\mathbb{C}^{2N})$ of multiplication by $\mathbf{V}(x,\omega)$ has a sufficiently small norm; it is enough to have

$$\|\mathbf{V}\|_{L^2_{-s}\to L^2_s}\|\big(\mathbf{J}(\mathbf{D}_m-\omega)-z\big)^{-1}\|_{L^2_s\to L^2_{-s}}<1/2. \tag{7.6}$$

Due to the bound on the action (7.4), the inequality (7.6) is satisfied since

$$\lim_{\omega\to\omega_b}\|\mathbf{V}(\omega)\|_{L^2_{-s}(\mathbb{R}^n,\mathbb{C}^{2N})\to L^2_s(\mathbb{R}^n,\mathbb{C}^{2N})}\le\lim_{\omega\to\omega_b}\|\langle Q\rangle^{2s}\mathbf{V}(\omega)\|_{L^\infty(\mathbb{R}^n,\mathrm{End}(\mathbb{C}^{2N}))}=0$$

by the assumption of the theorem. □

Lemma 7.3. *Assume that there is $s>1/2$ such that*

$$\lim_{\omega\to\omega_b}\|\langle Q\rangle^{2s}\mathbf{V}(\omega)\|_{L^\infty(\mathbb{R}^n,\mathrm{End}(\mathbb{C}^{2N}))}=0.$$

If $\omega_j\to\omega_b\in\{\pm m\}$, $\lambda_j\in\sigma_p(\mathbf{JL}(\omega_j))$, $\mathrm{Re}\,\lambda_j\ne0$, then $\lim_{j\to\infty}\lambda_j=0$.

Proof. Since $\mathbf{JL}(\omega_b)=\mathbf{J}(\mathbf{D}_m-\omega_b)$, is a differential operator with constant coefficients and with nondegenerate principal symbol, it has no point spectrum. Then Lemma 6.2 provides the conclusion. □

Remark 7.3. Thus, bifurcations at $\omega_b=\pm m$ off the imaginary axis are not allowed. At the same time, bifurcations at $\omega_b=\pm m$ from $\lambda_b=\pm2m\mathrm{i}$ are possible and indeed take place for the linearizations at solitary waves: by Lemma 4.6, there are families of eigenvalues $\lambda=\pm2\omega\mathrm{i}\in\sigma_p(\mathbf{JL}(\omega))$.

Let $\mathbf{J}\in\mathrm{End}(\mathbb{C}^{2N})$ be skew-adjoint and invertible, and let

$$\mathbf{L}(\omega)=\mathbf{D}_m-\omega+\mathbf{V}(x,\omega),\qquad\omega\in[-m,m],$$

with $\mathbf{V}:\ [-m,m]\to L^\infty(\mathbb{R}^n,\mathrm{End}(\mathbb{C}^{2N}))$ an operator-valued function of $\omega\in[-m,m]$, with $\mathbf{V}(\omega)$ a zero-order operator which is self-adjoint for each $\omega\in[-m,m]$.

Lemma 7.4. *Let $\omega_b=\pm m$ and assume that there is $C<\infty$ such that*

$$\|\mathbf{V}(\omega)\|_{L^\infty(\mathbb{R}^n,\mathrm{End}(\mathbb{C}^{2N}))}\le C|\omega-\omega_b|,\qquad\omega\in[-m,m]. \tag{7.7}$$

Let $\omega_j\in(-m,m)$, $j\in\mathbb{N}$; $\omega_j\to\omega_b$. If there is a sequence $\lambda_j\in\sigma(\mathbf{JL}(\omega_j))$, such that $\mathrm{Re}\,\lambda_j\ne0$, $\lim\limits_{j\to\infty}\lambda_j=0$, then

$$\lambda_j=O(|\omega_j-\omega_b|).$$

Remark 7.4. Due to asymptotics (3.6), the condition (7.7) is satisfied for linearization at solitary waves.

Proof. Without loss of generality, we will assume that $\omega_b=m$. We have: $\mathbf{JL}(\omega_j)\boldsymbol{\zeta}_j=\lambda_j\boldsymbol{\zeta}_j$, $\boldsymbol{\zeta}_j\in L^2(\mathbb{R}^n,\mathbb{C}^{2N})$, and without loss of generality we assume that $\|\boldsymbol{\zeta}_j\|_{L^2}=1$. We write:

$$(\mathbf{D}_m-\omega_j+\mathbf{J}\lambda_j)\boldsymbol{\zeta}_j=-\mathbf{V}(\omega_j)\boldsymbol{\zeta}_j. \tag{7.8}$$

Let $\Pi^\pm$ be orthogonal projections onto eigenspaces of $\mathbf{J}$ corresponding to $\pm\mathrm{i}\in\sigma(\mathbf{J})$. Applying $\Pi^\pm$ to (7.8) and denoting $\boldsymbol{\zeta}_j^\pm=\Pi^\pm\boldsymbol{\zeta}_j$, we get:

$$(\mathbf{D}_m-\omega_j+\mathrm{i}\lambda_j)\boldsymbol{\zeta}_j^+=-\Pi^+\mathbf{V}(\omega_j)\boldsymbol{\zeta}_j,\qquad(\mathbf{D}_m-\omega_j-\mathrm{i}\lambda_j)\boldsymbol{\zeta}_j^-=-\Pi^-\mathbf{V}(\omega_j)\boldsymbol{\zeta}_j. \tag{7.9}$$

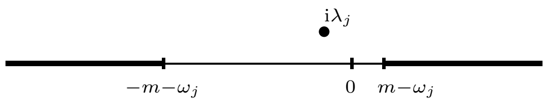

Figure 1: The closest point from $\sigma(\mathbf{D}_m - \omega_j)$ to $\mathrm{i}\lambda_j$ is $m - \omega_j$.

Since $\omega_j \to \omega_b = m$, without loss of generality, we can assume that $\omega_j > m/2$ for all $j \in \mathbb{N}$. Since the spectrum $\sigma(\mathrm{JL})$ is symmetric with respect to real and imaginary axes, we may assume, without loss of generality, that $\operatorname{Im}\lambda_j \geq 0$ for all $j \in \mathbb{N}$, so that $\operatorname{Re}\mathrm{i}\lambda_j \leq 0$ (see Figure 1). At the same time, since $\lambda_j \to 0$, we can assume that $|\lambda_j| \leq m/2$.

With $\mathbf{D}_m - \omega_j$ being self-adjoint, one has

$$\|(\mathbf{D}_m - \omega_j - \mathrm{i}\lambda_j)^{-1}\| = \frac{1}{\operatorname{dist}(\mathrm{i}\lambda_j, \sigma(\mathbf{D}_m - \omega_j))} = \frac{1}{|m - \omega_j - \mathrm{i}\lambda_j|}. \tag{7.10}$$

Remark 7.5. If $\operatorname{Re}\mathrm{i}\lambda_j > m - \omega_j$, then $\operatorname{dist}(\mathrm{i}\lambda_j, \sigma(\mathbf{D}_m - \omega_j)) < |m - \omega_j - \mathrm{i}\lambda_j|$, and the estimate (7.10) does not hold.

Combining (7.9) and (7.10), we get

$$\|\boldsymbol{\zeta}_j^-\|_{L^2} \leq \frac{\|\Pi^- \mathbf{V}\boldsymbol{\zeta}_j\|}{|m - \omega_j - \mathrm{i}\lambda_j|} \leq \frac{C(m^2 - \omega_j^2)}{|m - \omega_j - \mathrm{i}\lambda_j|}. \tag{7.11}$$

We used the normalization $\|\boldsymbol{\zeta}_j\| = 1$ and the bound $\|\Pi^- \mathbf{V}(\omega_j)\|_{L^2 \to L^2} \leq C|\omega_j - \omega_b|$ (cf. (7.7)). At the same time, due to $\operatorname{Re}\lambda_j \neq 0$, Lemma 6.1 yields $0 = \langle \boldsymbol{\zeta}_j, \mathbf{J}\boldsymbol{\zeta}_j \rangle = \mathrm{i}\|\boldsymbol{\zeta}_j^+\|_{L^2}^2 - \mathrm{i}\|\boldsymbol{\zeta}_j^-\|_{L^2}^2$,

$$\|\boldsymbol{\zeta}_j^+\|^2 = \|\boldsymbol{\zeta}_j^-\|^2 = \frac{1}{2}\|\boldsymbol{\zeta}_j\|^2 = \frac{1}{2};$$

then (7.11) yields

$$|m - \omega_j - \mathrm{i}\lambda_j| \leq \sqrt{2}C(m^2 - \omega_j^2),$$

leading to

$$|\lambda_j| \leq \sqrt{2}C(m^2 - \omega_j^2) + |m - \omega_j| \leq \big(\sqrt{2}C + \frac{1}{2m}\big)(m^2 - \omega_j^2). \qquad \square$$

Lemmas 6.2, 6.3, 7.2, 7.3, 7.4 complete the proof of Theorem 2.3.

8 Bifurcations from the origin

Here we prove Theorem 2.4, which we rewrite for the operator $\mathbf{JL}(\omega)$ which was defined in (4.3).

Lemma 8.1. *Let $n \leq 3$, and let $f \in C^\infty(\mathbb{R})$, $f(\eta) = \eta^k + o(\eta^k)$, $k \in \mathbb{N}$. If $n = 3$, further assume that $k = 1$. Let $\phi_\omega e^{-\mathrm{i}\omega t}$ be solitary waves from Lemma 3.5. Let $\omega_j \to m$, and let $\lambda_j \in \sigma_p(\mathbf{JL}(\omega_j))$, $\lambda_j = O(m^2 - \omega_j^2)$. Assume that the operator $\mathfrak{l}_-$ introduced in* (2.7) *satisfies $\sigma_p(\mathfrak{l}_-) = \{0\}$.*

1. *Assume that $\Lambda = 0$ is an eigenvalue of $\mathbf{j}\mathfrak{l}$ of algebraic multiplicity $2 + 2n$ (either $n = 1$, $k \neq 2$; or $n = 2$, $k > 1$; or $n = 3$). Then $\lambda_j \neq 0$ for $j \in \mathbb{N}$ implies that*

$$\Lambda_b := \lim_{j \to \infty} \frac{\lambda_j}{m^2 - \omega_j^2} \neq 0.$$

2. *$\Lambda_b \in \sigma_p(\mathbf{j}\mathfrak{l})$ or $\Lambda_b \in \sigma_p(\mathfrak{l}_-)$.*

3. *If $\operatorname{Re}\lambda_j \neq 0$, $j \in \mathbb{N}$, then Λ_b has zero Krein signature: a corresponding eigenvector $\boldsymbol{z}$ satisfies*

$$\langle \boldsymbol{z}, \mathfrak{l}\boldsymbol{z} \rangle = 0.$$

The rest of this section is devoted to the proof of Lemma 8.1.

Rescaled system

Let $\boldsymbol{\zeta}_j$ be the eigenfunctions corresponding to λ_j. Below, usually we will not write the subscripts j. The eigenfunction $\boldsymbol{\zeta}_j$ corresponding to $\lambda_j \in \sigma_p(\mathbf{J}\mathbf{L}(\omega))$ satisfies

$$\mathbf{D}_m \boldsymbol{\zeta}_j - \omega \boldsymbol{\zeta}_j + \lambda_j \mathbf{J} \boldsymbol{\zeta}_j + \mathbf{V}(\omega_j)\boldsymbol{\zeta}_j = 0, \tag{8.1}$$

where

$$\mathbf{V}(\omega)\boldsymbol{\zeta} = -f(\boldsymbol{\phi}_\omega^* \boldsymbol{\beta} \boldsymbol{\phi}_\omega)\boldsymbol{\beta}\boldsymbol{\zeta} - 2f'(\boldsymbol{\phi}_\omega^* \boldsymbol{\beta} \boldsymbol{\phi}_\omega)(\boldsymbol{\phi}_\omega^* \boldsymbol{\beta} \boldsymbol{\zeta})\boldsymbol{\beta}\boldsymbol{\phi}_\omega. \tag{8.2}$$

Let $\epsilon_j = \sqrt{m^2 - \omega_j^2}$. Denote $\Lambda_j = \lambda_j/\epsilon_j^2$. Let $\Pi^\pm$ be the projections corresponding to $\pm \mathrm{i} \in \sigma(\mathbf{J})$, and let Π_P, Π_A be the projections corresponding to $\pm 1 \in \sigma(\boldsymbol{\beta})$. We denote the "particle" and "antiparticle" components of $\boldsymbol{\zeta}_j$ by the relations

$$\Pi_P \boldsymbol{\zeta}_j(x) = \mathbf{P}_j(\epsilon_j x, \epsilon_j), \qquad \Pi_A \boldsymbol{\zeta}_j(x) = \epsilon \mathbf{A}_j(\epsilon_j x, \epsilon_j).$$

We also denote $\Pi^\pm \Pi_P = \Pi_P^\pm$, $\Pi^\pm \Pi_A = \Pi_A^\pm$,

$$\Pi_P^\pm \boldsymbol{\zeta}_j(x) = \mathbf{P}_j^\pm(\epsilon_j x, \epsilon_j), \qquad \Pi_A^\pm \boldsymbol{\zeta}_j(x) = \epsilon \mathbf{A}_j^\pm(\epsilon_j x, \epsilon_j).$$

Then

$$\mathbf{D}_0 \Pi_P^\pm \boldsymbol{\zeta}_j(x, \omega_j) = \epsilon_j (\mathbf{D}_0 \mathbf{P}_j^\pm)(\epsilon_j x, \epsilon_j), \qquad \mathbf{D}_0 \Pi_A^\pm \boldsymbol{\zeta}_j(x, \omega_j) = \epsilon_j (\mathbf{D}_0 \mathbf{A}_j^\pm)(\epsilon_j x, \epsilon_j).$$

Let $\boldsymbol{W} \in C^1\big(\mathbb{R}^n \times (0, \epsilon_0), \mathrm{End}(\mathbb{C}^{2N})\big)$ be such that $\mathbf{V}(x, \omega) = \epsilon^2 \boldsymbol{W}(\epsilon x, \epsilon)$.

Applying projections $\Pi_P^\pm$, $\Pi_A^\pm$ to (8.1) and dividing by ϵ^2 in the former case and by ϵ in the latter, we obtain the following system:

$$\mathbf{D}_0 \mathbf{A}_j^\pm + \Big(\frac{1}{m+\omega} \pm \mathrm{i}\Lambda_j\Big)\mathbf{P}_j^\pm + \Pi_P^\pm \boldsymbol{W}(y, \epsilon)(\mathbf{P}_j + \epsilon \mathbf{A}_j) = 0, \tag{8.3}$$

$$\mathbf{D}_0 \mathbf{P}_j^\pm + \Big(-m - \omega \pm \epsilon^2 \mathrm{i}\Lambda_j\Big)\mathbf{A}_j^\pm + \epsilon \Pi_A^\pm \boldsymbol{W}(y, \epsilon)(\mathbf{P}_j + \epsilon \mathbf{A}_j) = 0. \tag{8.4}$$

We took into account the relation $\mathbf{D}_0 \Pi_A^\pm = \Pi_P^\pm \mathbf{D}_0$.

We use (8.4) to express

$$\mathbf{A}_j^\pm = \frac{\mathbf{D}_0 \mathbf{P}_j^\pm + \epsilon \Pi_A^\pm \boldsymbol{W}(\mathbf{P}_j + \epsilon \mathbf{A}_j)}{m + \omega \mp \epsilon^2 \mathrm{i}\Lambda_j}, \qquad \mathbf{A}_j = \mathbf{A}_j^+ + \mathbf{A}_j^-, \tag{8.5}$$

and substitute this into (8.3):

$$-\frac{\Delta \mathbf{P}_j^\pm}{m + \omega - \epsilon^2 \mathrm{i}\Lambda_j} + \Big(\frac{1}{m+\omega} \pm \mathrm{i}\Lambda_j\Big)\mathbf{P}_j^\pm + \Pi_P^\pm \Big[\boldsymbol{W}(\mathbf{P}_j + \epsilon \mathbf{A}_j) + \epsilon \frac{(\mathbf{D}_0 \boldsymbol{W})(\mathbf{P}_j + \epsilon \mathbf{A}_j)}{m + \omega \mp \epsilon^2 \mathrm{i}\Lambda_j}\Big] = 0. \tag{8.6}$$

Denoting

$$\mathbf{Q}_j = \mathbf{P}_j^+ - \mathbf{P}_j^-$$

and taking the sum and the difference of (8.6), we have:

$$-\frac{\Delta \mathbf{P}_j}{m + \omega - \epsilon^2 \mathrm{i}\Lambda_j} + \frac{\mathbf{P}_j}{m + \omega} + \mathrm{i}\Lambda_j \mathbf{Q}_j + \Pi_P \Big[\boldsymbol{W}(\mathbf{P}_j + \epsilon \mathbf{A}_j) + \epsilon \frac{(\mathbf{D}_0 \boldsymbol{W})(\mathbf{P}_j + \epsilon \mathbf{A}_j)}{m + \omega \mp \epsilon^2 \mathrm{i}\Lambda_j}\Big] = 0, \tag{8.7}$$

$$-\frac{\Delta \mathbf{Q}_j}{m + \omega - \epsilon^2 \mathrm{i}\Lambda_j} + \frac{\mathbf{Q}_j}{m + \omega} + \mathrm{i}\Lambda_j \mathbf{P}_j + (\Pi_P^+ - \Pi_P^-)\Big[\boldsymbol{W}(\mathbf{P}_j + \epsilon \mathbf{A}_j) + \epsilon \frac{(\mathbf{D}_0 \boldsymbol{W})(\mathbf{P}_j + \epsilon \mathbf{A}_j)}{m + \omega \mp \epsilon^2 \mathrm{i}\Lambda_j}\Big] = 0. \tag{8.8}$$

The relative compactness of the eigenspaces

Lemma 8.2. *For any $s \in \mathbb{R}$, the multiplication operator*

$$\mathsf{Z} \mapsto \mathsf{W}_1^{\pm}\mathsf{Z} = \mathsf{W}\mathsf{Z} + \frac{(\mathbf{D}_0 \boldsymbol{W})\mathsf{Z}}{m + \omega \mp \epsilon^2 \mathrm{i}\Lambda}$$

is bounded from L^2_{-s} to L^2_s.

Proof. This follows from $\boldsymbol{W}$, $\nabla_y \boldsymbol{W}$ being exponentially decaying, as functions of $y = \epsilon x$, uniformly as $\epsilon \to 0$, due to Lemma 3.5 and due to the exponential decay of the solution u_k to (2.6); see [BL83]. □

Without loss of generality, we assume that $\operatorname{Im} \Lambda_b \le 0$. Applying [Agm75, Theorem 3.2] to (8.6), we conclude that, for any $s > 1/2$,

$$\|\mathbf{P}_j^-\|_{H^2_{s-1}} \le C \|\boldsymbol{W}_1(\mathbf{P}_j + \epsilon \mathbf{A}_j)\|_{L^2_s} \le C \|(\mathbf{P}_j + \epsilon \mathbf{A}_j)\|_{L^2_{-s}}, \tag{8.9}$$

for some $C < \infty$ which does not depend on $j \in \mathbb{N}$. Let us choose the normalization of $\boldsymbol{\zeta}_j$ so that

$$\sum_{\pm} (\|\mathbf{P}_j^{\pm}\|^2_{L^2} + \epsilon^2 \|\mathbf{A}_j^{\pm}\|^2_{L^2}) = 1, \qquad j \in \mathbb{N}. \tag{8.10}$$

Therefore, the terms in the right-hand side of (8.9) are bounded in L^2_s, and this leads to

$$\|\mathbf{P}_j^-\|_{H^2_{s-1}} \le C, \tag{8.11}$$

for some C independent of $j \in \mathbb{N}$. Using (8.10) and (8.11) in (8.5), we also get the bound for $\|\mathbf{A}_j^-\|_{H^1_{s-1}}$ (as long as $\epsilon > 0$ is sufficiently small). Thus, $\begin{bmatrix} \mathbf{P}_j^- \\ \mathbf{A}_j^- \end{bmatrix}$ is bounded in $H^1_{s-1}(\mathbb{R}^n, \mathbb{C}^{2N})$. Let $s > 1$; then the embedding $H^1_{s-1}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is compact, and there is a subsequence of $\begin{bmatrix} \mathbf{P}_j^- \\ \mathbf{A}_j^- \end{bmatrix}$ convergent to some $\begin{bmatrix} \hat{\mathbf{P}}^- \\ \hat{\mathbf{A}}^- \end{bmatrix} \in H^1_{s-1}(\mathbb{R}^n, \mathbb{C}^{2N})$, weakly in H^1_{s-1} and strongly in L^2. Let us prove that this limit is different from zero. By Lemma 6.1, $\|\Pi_+ \boldsymbol{\zeta}_j\|_{L^2} = \|\Pi_- \boldsymbol{\zeta}_j\|_{L^2}$, hence

$$\|\mathbf{P}_j^+\|^2_{L^2} + \epsilon^2 \|\mathbf{A}_j^+\|^2_{L^2} = \|\mathbf{P}_j^-\|^2_{L^2} + \epsilon^2 \|\mathbf{A}_j^-\|^2_{L^2}; \tag{8.12}$$

If $\mathbf{P}_j^- \to 0$ in H^1, then, by (8.5), $\mathbf{A}_j^- \to 0$ in L^2, hence both sides of (8.12) converge to zero, contradicting (8.10).

Recall that $\hat{\Phi}_P = \boldsymbol{n} u_k$, where u_k is a positive spherically symmetric solution to (2.6) and $\boldsymbol{n} \in \mathbb{C}^N$ is such that $\beta \boldsymbol{n} = \boldsymbol{n}$, $\|\boldsymbol{n}\| = 1$; one has $\Pi_P \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix}$.

The limit system

Let $\hat{\boldsymbol{W}}(y) = \boldsymbol{W}(y, 0)$. Substituting $\lim_{\epsilon \to 0} \epsilon^{-1/(2k)} \phi_P(\epsilon^{-1} y) = \hat{\Phi}_P(y) = \boldsymbol{n} u_k(y)$, with $\boldsymbol{n} \in \mathbb{C}^N$, $|\boldsymbol{n}| = 1$, we derive:

$$\begin{aligned} \hat{\boldsymbol{W}}\mathbf{P} = \lim_{j \to \infty} \boldsymbol{W}(y, \epsilon_j)(\mathbf{P}_j + \epsilon_j \mathbf{A}_j) &= -|\hat{\Phi}_P|^{2k}\mathbf{P} - 2k|\hat{\Phi}_P|^{2k-2}(\hat{\Phi}_P^* \mathbf{P}) \begin{bmatrix} \hat{\Phi}_P \\ 0 \end{bmatrix} \\ &= -|\hat{\Phi}_P|^{2k}\mathbf{P} - 2k|\hat{\Phi}_P|^{2k} \langle \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix}, \mathbf{P} \rangle_{\mathbb{C}^{2N}} \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix}. \end{aligned} \tag{8.13}$$

Above, $\langle\, , \rangle_{\mathbb{C}^{2N}}$ is the inner product in $\mathbb{C}^{2N}$. Considering (8.6) in the limit $\epsilon \to 0$, we have

$$-\frac{\Delta \hat{\mathbf{P}}^{\pm}}{2m} + \frac{\hat{\mathbf{P}}^{\pm}}{2m} - u_k^{2k} \hat{\mathbf{P}}^{\pm} - 2k u_k^{2k} \Big\langle \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix}, \hat{\mathbf{P}} \Big\rangle_{\mathbb{C}^{2N}} \Pi^{\pm} \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix} \pm \mathrm{i}\Lambda_b \hat{\mathbf{P}}^{\pm} = 0, \tag{8.14}$$

where we used (8.13). We used the equality $\beta \boldsymbol{n} = \boldsymbol{n}$ which implies $\Pi_P \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix}$, Defining $\hat{\mathbf{Q}} := \hat{\mathbf{P}}^+ - \hat{\mathbf{P}}^-$, we get the following system:

$$-\frac{\Delta \hat{\mathbf{P}}}{2m} + \frac{\hat{\mathbf{P}}}{2m} - u_k^{2k} \hat{\mathbf{P}} - 2k u_k^{2k} \Big\langle \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix}, \hat{\mathbf{P}} \Big\rangle_{\mathbb{C}^{2N}} \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix} + \mathrm{i}\Lambda_b \hat{\mathbf{Q}} = 0, \tag{8.15}$$

$$
-\frac{\Delta\hat{\mathbf{Q}}}{2m}+\frac{\hat{\mathbf{Q}}}{2m}-u_k^{2k}\hat{\mathbf{Q}}-2ku_k^{2k}\Big\langle\begin{bmatrix}\boldsymbol{n}\\0\end{bmatrix},\hat{\mathbf{P}}\Big\rangle_{\mathbb{C}^{2N}}\begin{bmatrix}0\\\mathrm{i}\boldsymbol{n}\end{bmatrix}+\mathrm{i}\Lambda_b\hat{\mathbf{P}}=0. \tag{8.16}
$$

We used the relation $(\Pi^+-\Pi^-)\begin{bmatrix}\boldsymbol{n}\\0\end{bmatrix}=-\mathrm{i}\mathbf{J}\begin{bmatrix}\boldsymbol{n}\\0\end{bmatrix}=\begin{bmatrix}0\\\mathrm{i}\boldsymbol{n}\end{bmatrix}$.

Proof of Lemma 8.1 *(1)*

We consider the system (8.7), (8.8) (with the expression (8.5) for $\boldsymbol{A}_j$) as a perturbation of the system (8.15), (8.16). By [Com11, Lemma 4.1], we have:

$$
\mathscr{N}_g(\mathbf{JL}(\omega))\supset\mathrm{Span}\{\mathbf{J}\boldsymbol{\phi},\,\partial_\omega\boldsymbol{\phi},\,\partial_{x^\jmath}\boldsymbol{\phi},\,(\boldsymbol{\alpha}^\jmath-2\omega x^\jmath\mathbf{J})\boldsymbol{\phi}\ ;\ 1\le\jmath\le n\},\qquad\dim\mathscr{N}_g(\mathbf{JL}(\omega))\ge 2n+2.
$$

If there is an eigenvalue family $(\lambda_j)_(j\in\mathbb{N})$ such that $\Lambda_j\ne0$, $\Lambda_j=\frac{\lambda_j}{m^2-\omega_j^2}\to0$, converging to zero as $\omega_j\to m$, then we would have

$$
\dim\mathscr{N}_g(\mathbf{j}\mathfrak{l})\ge\dim\mathscr{N}_g(\mathbf{JL}(\omega))|_{\omega<m}+1\ge 2n+3.
$$

We conclude that if $\dim\mathscr{N}_g(\mathbf{j}\mathfrak{l})=2n+2$ and $\Lambda_j\to0$, then $\Lambda_j\equiv0$ for all j sufficiently large.

Proof of Lemma 8.1 *(2)*

In the case when $p(y):=\Big\langle\begin{bmatrix}\boldsymbol{n}\\0\end{bmatrix},\hat{\mathbf{P}}(y)\Big\rangle_{\mathbb{C}^{2N}}$ is identically zero, (8.14) implies that either $\mathrm{i}\Lambda_b$ or $-\mathrm{i}\Lambda_b$ belongs to $\sigma_p(\mathfrak{l}_-)$, with the eigenfunction being (either nonzero component of) $\hat{\mathbf{P}}^\pm\in L^2(\mathbb{R}^n,\mathbb{C}^{2N})$. Since we assumed that $\sigma_p(\mathfrak{l}_-)=\{0\}$, we conclude that $\Lambda_b=0$. Let us note that (8.15) implies that in this case $\hat{\mathbf{P}}(y)=u_k(y)\boldsymbol{M}$, for some $\boldsymbol{M}\in\mathbb{C}^{2N}$. Similarly, (8.16) implies that $\hat{\mathbf{Q}}(y)=u_k(y)\boldsymbol{N}$, for some $\boldsymbol{N}\in\mathbb{C}^{2N}$.

Let us now consider the case when $p\in L^2(\mathbb{R}^n,\mathbb{C})$ is not identically zero. Taking the inner product of (8.15), (8.16) with $\begin{bmatrix}\boldsymbol{n}\\0\end{bmatrix}$ and denoting

$$
p=\Big\langle\begin{bmatrix}\boldsymbol{n}\\0\end{bmatrix},\hat{\mathbf{P}}\Big\rangle_{\mathbb{C}^{2N}},\qquad q=\Big\langle\begin{bmatrix}\boldsymbol{n}\\0\end{bmatrix},\hat{\mathbf{Q}}\Big\rangle_{\mathbb{C}^{2N}}, \tag{8.17}
$$

we get the equations

$$
\mathfrak{l}_-p-2ku_k^{2k}p+\mathrm{i}\Lambda_bq=\mathfrak{l}_+p+\mathrm{i}\Lambda_bq=0,\qquad\mathfrak{l}_-q+\mathrm{i}\Lambda_bp=0, \tag{8.18}
$$

which we write as

$$
\begin{bmatrix}0&\mathfrak{l}_-\\-\mathfrak{l}_+&0\end{bmatrix}\begin{bmatrix}p\\\mathrm{i}q\end{bmatrix}=\Lambda_b\begin{bmatrix}p\\\mathrm{i}q\end{bmatrix}. \tag{8.19}
$$

This relation implies that Λ_b is the eigenvalue of the linearization of the nonlinear Schrödinger equation

$$
\mathrm{i}\dot{\psi}=-\frac{1}{2m}\Delta\psi-|\psi|^{2k}\psi,\qquad\psi(x,t)\in\mathbb{C},\qquad x\in\mathbb{R}^n
$$

at the solitary wave $u_k(x)e^{-\mathrm{i}\omega t}$, $\omega=-\frac{1}{2m}$ (cf. (1.6), (1.7)).

Proof of Lemma 8.1 *(3)*

If $(p,\,q)\in L^2(\mathbb{R}^n,\mathbb{C})\times L^2(\mathbb{R}^n,\mathbb{C})$ satisfies (8.19) and $p=\Big\langle\begin{bmatrix}\boldsymbol{n}\\0\end{bmatrix},\hat{\mathbf{P}}\Big\rangle_{\mathbb{C}^{2N}}$, then

$$
\tilde{\mathbf{P}}=\begin{bmatrix}p\boldsymbol{n}\\\mathrm{i}q\boldsymbol{n}\end{bmatrix},\qquad\tilde{\mathbf{Q}}=\begin{bmatrix}q\boldsymbol{n}\\\mathrm{i}p\boldsymbol{n}\end{bmatrix}
$$

solve (8.15), (8.16). It follows that $\hat{\mathbf{P}}-\tilde{\mathbf{P}}$, $\hat{\mathbf{Q}}-\tilde{\mathbf{Q}}$ satisfy

$$
\begin{bmatrix}0&\mathfrak{l}_-\otimes I_{\mathbb{C}^{2N}}\\-\mathfrak{l}_-\otimes I_{\mathbb{C}^{2N}}&0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{P}}-\tilde{\mathbf{P}}\\\mathrm{i}(\hat{\mathbf{Q}}-\tilde{\mathbf{Q}})\end{bmatrix}=\Lambda_b\begin{bmatrix}\hat{\mathbf{P}}-\tilde{\mathbf{P}}\\\mathrm{i}(\hat{\mathbf{Q}}-\tilde{\mathbf{Q}})\end{bmatrix}. \tag{8.20}
$$

In the case $\Lambda_b = 0$, by (8.19), $\boldsymbol{z} = \begin{bmatrix} p \\ \mathrm{i}q \end{bmatrix}$ satisfies $\mathfrak{l}\boldsymbol{z} = 0$, leading to $\langle \boldsymbol{z}, \mathfrak{l}\boldsymbol{z}\rangle = 0$. Then $\hat{\mathbf{P}} - \tilde{\mathbf{P}} = u_k \boldsymbol{M}$, $\hat{\mathbf{Q}} - \tilde{\mathbf{Q}} = u_k \boldsymbol{N}$ for some $\boldsymbol{M}$, $\boldsymbol{N} \in \mathbb{C}^{2N}$. As follows from (8.19), $p \in \ker \mathfrak{l}_+ = \mathrm{Span}\{\partial_j u_k \ ; \ 1 \le j \le n\}$, while $q \in \ker \mathfrak{l}_- = \mathrm{Span}\{u_k\}$. For $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$, we have:

$$\hat{\mathbf{P}} = \begin{bmatrix} p\boldsymbol{n} \\ \mathrm{i}q\boldsymbol{n} \end{bmatrix} + \boldsymbol{M} u_k, \qquad \hat{\mathbf{Q}} = \begin{bmatrix} q\boldsymbol{n} \\ \mathrm{i}p\boldsymbol{n} \end{bmatrix} + \boldsymbol{N} u_k. \tag{8.21}$$

Since $p = \left\langle \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix}, \hat{\mathbf{P}} \right\rangle_{\mathbb{C}^{2N}}$, $\boldsymbol{M}$ is to satisfy $\left\langle \begin{bmatrix} \boldsymbol{n} \\ 0 \end{bmatrix}, \boldsymbol{M} \right\rangle_{\mathbb{C}^{2N}} = 0$. $\boldsymbol{N}$ is to satisfy $\boldsymbol{N} = \Pi^+ \boldsymbol{M} - \Pi^- \boldsymbol{M}$.

Now let us consider the case when $\Lambda_b \ne 0$. Since $\sigma_p(\mathfrak{l}_-) = \{0\}$, we conclude that $\hat{\mathbf{P}} = \tilde{\mathbf{P}}$, $\hat{\mathbf{Q}} = \tilde{\mathbf{Q}}$. Thus,

$$\hat{\mathbf{P}} = \begin{bmatrix} p\boldsymbol{n} \\ \mathrm{i}q\boldsymbol{n} \end{bmatrix}, \qquad \hat{\mathbf{Q}} = \begin{bmatrix} q\boldsymbol{n} \\ \mathrm{i}p\boldsymbol{n} \end{bmatrix}. \tag{8.22}$$

Assume that the sequence $\lambda_j \in \sigma_p(\mathbf{JL})$ satisfies $\operatorname{Re}\lambda_j \ne 0$. By Lemma 6.1, the corresponding eigenvectors satisfy

$$0 = \langle \boldsymbol{\zeta}_j, \mathbf{J}\boldsymbol{\zeta}_j \rangle = \langle \mathbf{P}_j + \epsilon \mathbf{A}_j, \mathbf{J}(\mathbf{P}_j + \epsilon \mathbf{A}_j)\rangle;$$

this leads in the limit $\epsilon \to 0$ to

$$0 = \langle \hat{\mathbf{P}}, \mathbf{J}\hat{\mathbf{P}} \rangle = \mathrm{i}\|\hat{\mathbf{P}}^+\|^2 - \mathrm{i}\|\hat{\mathbf{P}}^-\|^2. \tag{8.23}$$

By (8.22),

$$\hat{\mathbf{P}} = \begin{bmatrix} p \\ \mathrm{i}q \end{bmatrix}; \qquad \hat{\mathbf{P}}^+ = \frac{1}{2}\begin{bmatrix} (p+q)\boldsymbol{n} \\ \mathrm{i}(p+q)\boldsymbol{n} \end{bmatrix}, \qquad \hat{\mathbf{P}}^- = \frac{1}{2}\begin{bmatrix} (p-q)\boldsymbol{n} \\ -\mathrm{i}(p-q)\boldsymbol{n} \end{bmatrix}.$$

The condition (8.23) leads to

$$0 = \|\hat{\mathbf{P}}^+\|^2 - \|\hat{\mathbf{P}}^-\|^2 = \frac{\|p\|^2 + \|q\|^2 + 2\operatorname{Re}\langle p, q\rangle}{2} - \frac{\|p\|^2 + \|q\|^2 - 2\operatorname{Re}\langle p, q\rangle}{2} = 2\operatorname{Re}\langle p, q\rangle. \tag{8.24}$$

By (8.19), the eigenvector of $\mathfrak{j}\mathfrak{l}$ corresponding to Λ_b is given by $\boldsymbol{z} = \begin{bmatrix} p \\ \mathrm{i}q \end{bmatrix}$; using (8.24), we compute:

$$\langle \boldsymbol{z}, \mathfrak{j}\boldsymbol{z}\rangle = \left\langle \begin{bmatrix} p \\ \mathrm{i}q \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p \\ \mathrm{i}q \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} p \\ \mathrm{i}q \end{bmatrix}, \begin{bmatrix} \mathrm{i}q \\ -p \end{bmatrix} \right\rangle = 2\mathrm{i}\operatorname{Re}\langle p, q\rangle = 0.$$

Then the relation $\mathfrak{l}\boldsymbol{z} + \Lambda_b \mathfrak{j}\boldsymbol{z} = 0$ leads to $\langle \boldsymbol{z}, \mathfrak{l}\boldsymbol{z}\rangle + \Lambda_b \langle \boldsymbol{z}, \mathfrak{j}\boldsymbol{z}\rangle = 0$, hence $\langle \boldsymbol{z}, \mathfrak{l}\boldsymbol{z}\rangle = 0$, completing the proof of Lemma 8.1 and thus of Theorem 2.4.

A Appendix: Hardy-type inequalities

Our aim in this section is to establish the generalized Hardy estimates used in the course of the study.

A.1 Hardy inequality in one dimension

In dimension 1, Hardy's inequality reads for an integrable function f, as

$$\int_0^\infty \frac{1}{x^2}\Big(\int_0^x |f(t)|^2\,dt\Big) \le 4\int_0^\infty |f(x)|^2\,dx. \tag{A.1}$$

The result above can be has the following generalization.

Lemma A.1 (Generalized Hardy inequality)**.** *Suppose that* $w \in C^1((0,\infty))$ *is a differentiable function with positive derivative on* $(0,+\infty)$ *and* $f \in C^1_{comp}((0,\infty),\mathbb{C})$. *Then*

$$\int_0^\infty w'|f|^2\,dx \le 4\int_0^\infty \frac{w^2}{w'}|f'|^2\,dx.$$

Proof. Without loss of generality, we can assume that $A = 0$. The integration by parts yields

$$\int_{\mathbb{R}+} w'(x)|f(x)|^2\,dx = -\int_{\mathbb{R}+} w(x)\overline{f}(x)f'(x)\,dx - \int_{\mathbb{R}+} w(x)\overline{f'}(x)f(x)\,dx.$$

Using the Cauchy-Schwarz inequality, we get:

$$\int_{\mathbb{R}+} w'(x)|f(x)|^2\,dx \le 2\sqrt{\int_{\mathbb{R}+} w'(x)|f(x)|^2\,dx}\sqrt{\int_{\mathbb{R}+} \frac{w(x)^2}{w'(x)}|f'(x)|^2\,dx},$$

and thus $\int_{\mathbb{R}+} w'(x)|f(x)|^2\,dx \le 4\int_{\mathbb{R}+} \frac{w^2(x)}{w'(x)}|f'(x)|^2\,dx$. $\square$

Let $\mathcal{H}^0_\varphi$ and $\mathcal{H}^1_\varphi$ be defined by

$$\mathcal{H}^0_\varphi := \left\{u \in L^2(\mathbb{R},\mathbb{C}^2)\ ;\ e^\varphi u \in L^2(\mathbb{R},\mathbb{C}^2)\right\}, \qquad \mathcal{H}^1_\varphi := \left\{u \in \mathcal{H}^0_\varphi\ ;\ \partial_x u \in \mathcal{H}^0_\varphi\right\},$$

endowed with the associated weighted L^2 and H^1 norms. The notation φ' stands for the radial derivative of φ. We will use the following elementary estimate.

Lemma A.2. *Let $N = 2l$, $l \in \mathbb{N}$. Let $m > 0$ and let α, β be self-adjoint matrices such that $\alpha^2 = \beta^2 = I_N$, $\alpha\beta + \beta\alpha = 0$. Then*

$$\|e^{\mathrm{i}\alpha(m\beta-\lambda)x}\|_{\mathbb{C}^N\to\mathbb{C}^N} \le C_1(\lambda) := \sqrt{\frac{|\lambda|+m}{|\lambda|-m}}, \qquad x \in \mathbb{R}, \qquad \lambda \in \mathbb{R}\setminus[-m,m].$$

There is $\varkappa = \varkappa(m) > 0$ such that

$$\|e^{\mathrm{i}\alpha(m\beta-\lambda)x}\|_{\mathbb{C}^N\to\mathbb{C}^N} \le \varkappa\langle x\rangle, \qquad x \in \mathbb{R}, \qquad \lambda \in \mathbb{R}\setminus(-m,m). \tag{A.2}$$

Proof. Let us notice that the matrix $\mathrm{i}\alpha(m\beta-\lambda)$ is skew-adjoint. Its eigenvalues are purely imaginary as long as $\lambda \in \mathbb{R}\setminus[-m,m]$. The norm of $e^{\mathrm{i}\alpha(m\beta-\lambda)}$ is larger than 1 because the projections onto eigenspaces are not orthogonal. There are 2×2 Jordan blocks when $\lambda = \pm m$, leading to the factor of $\langle x\rangle$ in (A.2).

Let us derive more careful estimates. Without loss of generality, we may take $\alpha = -\sigma_2$, $\beta = \sigma_3$; then

$$M(\lambda) := \mathrm{i}\alpha(m\beta-\lambda) = \begin{bmatrix} 0 & m+\lambda \\ m-\lambda & 0 \end{bmatrix}.$$

The eigenvalues of $M(\lambda)$ are $\pm\mathrm{i}\xi(\lambda)$, where $\xi(\lambda) = \sqrt{\lambda^2-m^2} \ge 0$. For $|\lambda| > m$, the operator $e^{-M(\lambda)x}$ is the operator of multiplication by a matrix-valued function, with the $\mathbb{C}^2\to\mathbb{C}^2$ norm bounded uniformly in x. Without loss of generality, let us assume that $\lambda > 0$. Then the eigenvectors of $M(\lambda)$ are given by $u_\pm = \begin{bmatrix} 1 \\ \pm\mathrm{i}\sqrt{\frac{\lambda-m}{\lambda+m}} \end{bmatrix}$. We note that for $A : \mathbb{C}^2\to\mathbb{C}^2$ with both eigenvalues of of magnitude 1,

$$\sup_x \|e^{M(\lambda)x}\|_{\mathbb{C}^2\to\mathbb{C}^2} \le \sup_x \sup_{u,\,v} \frac{|e^{M(\lambda)x}(u+v)|}{|u+v|} \le \sup_{u,\,v}\sqrt{\frac{|u|^2+|v|^2+2|\langle u,v\rangle|}{|u|^2+|v|^2-2|\langle u,v\rangle|}} \le \sup_{u,\,v}\sqrt{\frac{1+\frac{|\langle u,v\rangle|}{|u||v|}}{1-\frac{|\langle u,v\rangle|}{|u||v|}}},$$

with $\sup_{u,v}$ taken over all pairs of eigenvectors corresponding to $\pm\mathrm{i}\xi(\lambda)$. Since $\frac{|\langle u_+,u_-\rangle|}{|u_+||u_-|} = \frac{1-\frac{\lambda-m}{\lambda+m}}{1+\frac{\lambda-m}{\lambda+m}} = \frac{m}{\lambda}$, one has

$$\sup_x \|e^{\mathrm{i}M(\lambda)x}\| \le \sqrt{\frac{\lambda+m}{\lambda-m}}.$$

One similarly derives the estimate in the case $\lambda < -m$.

In the case when $\lambda = \pm m$, $M(\lambda)$ is a Jordan block corresponding to the eigenvalue zero; then for $|\lambda| \ge m$ and any $x \in \mathbb{R}$ one has $\|e^{-M(\lambda)x}\|_{\mathbb{C}^2\to\mathbb{C}^2} \le \varkappa\langle x\rangle$, with some $\varkappa = \varkappa(m) > 0$. $\square$

Let Q denote the operator of multiplication

$$Q: \ u(x) \mapsto xu(x), \qquad u \in \mathcal{S}'(\mathbb{R}).$$

Lemma A.3. *Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a radially even differentiable function with non-negative derivative in the radial variable.*

If $\lambda \in \mathbb{R} \setminus [-m, m]$ and if $u \in L^2(\mathbb{R}, \mathbb{C}^2)$ and $(D_m - \lambda)u$ belongs to $\mathcal{H}^0_\varphi$, then $u \in \mathcal{H}^0_\varphi$ and

$$\|\langle Q\rangle^{-1} u\|_{\mathcal{H}^1_\varphi} \le c(\lambda)\|(D_m - \lambda)u\|_{\mathcal{H}^0_\varphi}.$$

If $\lambda \in \mathbb{R} \setminus (-m, m)$ and if $u \in L^2(\mathbb{R}, \mathbb{C}^2)$ and $\langle Q\rangle(D_m - \lambda)u$ belongs to $\mathcal{H}^0_\varphi$, then $u \in \mathcal{H}^0_\varphi$ and

$$\|\langle Q\rangle^{-1} u\|_{\mathcal{H}^1_\varphi} \le c(\lambda)\|\langle Q\rangle(D_m - \lambda)u\|_{\mathcal{H}^0_\varphi}.$$

Proof. First let us give the proof for the case when $u \in C^1(\mathbb{R}, \mathbb{C}^2)$ has compact support. Define

$$f(x) := \mathrm{i}\alpha(-\mathrm{i}\alpha\partial_x + m\beta - \lambda)u(x);$$

one has $f \in \mathcal{H}^0_\varphi(\mathbb{R}, \mathbb{C}^2)$ due to the compactness of its support, and we can write

$$\partial_x u + M(\lambda)u = f \tag{A.3}$$

with

$$M(\lambda) := \mathrm{i}\alpha\beta m - \mathrm{i}\alpha\lambda.$$

The function u, being a solution to (A.3), could be expressed as

$$u(x) = \begin{cases} \int_{-\infty}^x e^{-M(\lambda)(x-y)} f(y)\,dy, & x \le 0; \\ -\int_x^{+\infty} e^{-M(\lambda)(x-y)} f(y)\,dy, & x > 0. \end{cases} \tag{A.4}$$

If $|\lambda| > m$, multiplying (A.4) by the weight e^φ and using Lemma A.2 leads to

$$e^{\varphi(x)}|u(x)| \le C_1(\lambda) \begin{cases} \int_{-\infty}^x e^{\varphi(y)}|f(y)|\,dy, & x \le 0, \\ \int_x^{+\infty} e^{\varphi(y)}|f(y)|\,dy, & x > 0, \end{cases} \tag{A.5}$$

since $\varphi(x) \le \varphi(y)$ for $|x| \le |y|$. Above, $C_1(\lambda) = \sqrt{\frac{|\lambda|+m}{|\lambda|-m}}$ was introduced in Lemma A.2. From the generalized Hardy inequality (cf. Lemma A.1) with $w(x) = x$, it follows that

$$\int_{\mathbb{R}} e^{2\varphi(x)}|u(x)|^2\,dx \le C_1(\lambda)^2 \int_{\mathbb{R}} \left(\int_{-\infty}^x e^{\varphi(y)}|f(y)|\,dy\right)^2 dx \le 4C_1(\lambda)^2 \int_{\mathbb{R}} |x|^2 e^{2\varphi(x)}|f(x)|^2\,dx.$$

By Hardy's inequality (A.1),

$$\int_{\mathbb{R}^+} \frac{e^{2\varphi(x)}|u(x)|^2}{\langle x\rangle^2}\,dx \le C_1(\lambda)^2 \int_{\mathbb{R}^+} \frac{1}{\langle x\rangle^2}\Big(\int_x^{+\infty} e^{\varphi(y)}|f(y)|\,dy\Big)^2 dx \le 4C_1(\lambda)^2 \int_{\mathbb{R}^+} e^{2\varphi(x)}|f(x)|^2\,dx,$$

and similarly for the integral over $\mathbb{R}^-$. Thus,

$$\|\langle Q\rangle^{-1}u\|_{\mathcal{H}^0_\varphi} \le 2C_1(\lambda)\|f\|_{\mathcal{H}^0_\varphi}. \tag{A.6}$$

By (A.3), $\partial_x(\langle Q\rangle^{-1}u) = [\partial_x, \langle Q\rangle^{-1}]u + \langle Q\rangle^{-1}Mu + \langle Q\rangle^{-1}f$; since $\|\langle Q\rangle^2[\partial_x, \langle Q\rangle^{-1}]\|_{L^\infty} \le 1$ and $\|M(\lambda)\|_{\mathbb{C}^2\to\mathbb{C}^2} = |\xi(\lambda)| \le |\lambda|$,

$$\|\partial_x(\langle Q\rangle^{-1}u)\|_{\mathcal{H}^0_\varphi} \le (|\lambda| + 1)\|\langle Q\rangle^{-1}u\|_{\mathcal{H}^0_\varphi} + \|\langle Q\rangle^{-1}f\|_{\mathcal{H}^0_\varphi}.$$

Due to (A.6), $\|\partial_x(\langle Q\rangle^{-1}u)\|_{\mathcal{H}^0_\varphi} \le \big(2(|\lambda| + 1)C_1(\lambda) + 1\big)\|f\|_{\mathcal{H}^0_\varphi}$, hence

$$\|\langle Q\rangle^{-1}u\|_{\mathcal{H}^1_\varphi} \le \big(2(|\lambda| + 2)C_1(\lambda) + 1\big)\|f\|_{\mathcal{H}^0_\varphi}. \tag{A.7}$$

If $|\lambda| \geq m$, multiplying (A.4) by the weight e^{φ} and using Lemma A.2 leads to

$$e^{\varphi(x)}|u(x)| \leq \varkappa \begin{cases} \int_{-\infty}^{x} e^{\varphi(x)}\langle x-y\rangle |f(y)|\,dy, & x \leq 0 \\ \int_{x}^{+\infty} e^{\varphi(x)}\langle x-y\rangle |f(y)|\,dy, & x > 0 \end{cases}. \tag{A.8}$$

Taking into account that on each of the regions of integration one has $|y| \geq |x-y|$, one obtains:

$$e^{\varphi(x)}|u(x)| \leq 2\varkappa \begin{cases} \int_{-\infty}^{x} \langle y\rangle |f(y)|\,dy, & x \leq 0 \\ \int_{x}^{+\infty} \langle y\rangle |f(y)|\,dy, & x > 0 \end{cases}. \tag{A.9}$$

Similarly to the previous from the Hardy inequality, there is $c > 0$ such that

$$\|\langle Q\rangle^{-1}u\|_{\mathcal{H}^0_\varphi} \leq c\|\langle Q\rangle f\|_{\mathcal{H}^0_\varphi}.$$

Now let $u \in L^2(\mathbb{R},\mathbb{C}^2)$ and $\langle Q\rangle(D_m-\lambda)u \in \mathcal{H}^0_\varphi(\mathbb{R},\mathbb{C}^2)$. We will focus on the case $|\lambda| > m$; the proof in the case $|\lambda| \geq m$ is similar.

First, assume that φ is bounded. If the assertion is true for C^1 functions, it is also true for square integrable function. Indeed, let η be the characteristic function of the unit ball, and set $\widehat{u_\epsilon}(\xi) = \eta(\epsilon\xi)\hat{u}(\xi)$, $\epsilon > 0$; then $u_\varepsilon \in C^1(\mathbb{R})$ such that $(D_m-\lambda)u_\varepsilon \in \mathcal{H}^0_\varphi(\mathbb{R})$ and $(D_m-\lambda)u_\varepsilon \to (D_m-\lambda)u$ in $\mathcal{H}^0_\varphi(\mathbb{R})$ as $\epsilon \to 0$.

In order to prove the assertion for C^1 functions, let us take a smooth cut-off function η which is symmetric, satisfies $0 \leq \eta \leq 1$, with support in $|x| \leq 2$, and is equal to 1 for $|x| \leq 1$. For $j \in \mathbb{N}$, define η_j by $\eta_j(x) := \eta(\frac{x}{j})$. Multiplying by η_j the relation

$$\partial_x u + M(\lambda)u =: f \in \mathcal{H}^0_\varphi(\mathbb{R},\mathbb{C}^2)$$

gives the following relation for $v_j = \eta_j u$:

$$\partial_x v_j + M(\lambda)v_j = \eta_j f + (\partial_x\eta_j)u \in \mathcal{H}^0_\varphi(\mathbb{R},\mathbb{C}^2).$$

Then, by (A.7), there is $c(\lambda) > 0$ such that

$$\|Q^{-1}\eta_j u\|_{\mathcal{H}^1_\varphi} \leq c(\lambda)\|\eta_j f + (\partial_x\eta_j)u\|_{\mathcal{H}^0_\varphi} \leq c(\lambda)\Big(\|f\|_{\mathcal{H}^0_\varphi} + \frac{1}{j}\|\nabla\eta\|_{L^\infty}\|\chi_{[j,2j]}u\|_{\mathcal{H}^0_\varphi}\Big),$$

as u is in L^2 and φ is bounded, we conclude by the dominated convergence theorem in the case when φ is bounded.

Now assume that φ is not necessarily bounded. Fix $x_0 > 0$. For $j \in \mathbb{N}$, define

$$\varphi_j(x) = \varphi(x_0) + \int_{x_0}^{x} \frac{\varphi'(y)}{\Big(1 + \frac{\varphi'(y)^2+y^2}{j}\Big)^2}\,dy.$$

Then $\varphi_j \nearrow \varphi$, $\varphi_j' \to \varphi'$, pointwise as $j \to \infty$. Moreover, $|\varphi_j(x)|$ is bounded, so that $\|u\|_{\mathcal{H}^0_{\varphi_j}} \leq c(\lambda)\|Qf\|_{\mathcal{H}^0_{\varphi_j}}$; then by the Fatou lemma for the left hand side and the dominated convergence theorem for the right hand side we conclude that

$$\|u\|_{\mathcal{H}^0_\varphi} \leq c(\lambda)\|Qf\|_{\mathcal{H}^0_\varphi}.$$

The estimates for the derivatives follows from (A.3). □

Lemma A.4. *Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a radially even differentiable function with derivative in the radial variable φ such that*

$$(\varphi')^2 < m^2 - \lambda^2.$$

If $\lambda \in (-m,m)$ and if $u \in L^2(\mathbb{R},\mathbb{C}^2)$ is such that $(D_m-\lambda)u$ belongs to $\mathcal{H}^0_\varphi$, then $u \in \mathcal{H}^0_\varphi$ and

$$\|\langle Q\rangle^{-1}u\|_{\mathcal{H}^1_\varphi} \leq c(\lambda)\|(D_m-\lambda)u\|_{\mathcal{H}^0_\varphi}.$$

If $\lambda \in [-m,m]$ and if $u \in L^2(\mathbb{R},\mathbb{C}^2)$ is such that $\langle Q\rangle(D_m-\lambda)u$ belongs to $\mathcal{H}^0_\varphi$, then $u \in \mathcal{H}^0_\varphi$ and

$$\|\langle Q\rangle^{-1}u\|_{\mathcal{H}^1_\varphi} \leq c(\lambda)\|\langle Q\rangle(D_m-\lambda)u\|_{\mathcal{H}^0_\varphi}.$$

Proof. The proof is almost the one of Lemma A.3. Starting from (A.3),

$$\partial_x u + M(\lambda)u = f.$$

If $\lambda \in (-m, m)$, diagonalizing the matrix $M(\lambda)$ gives

$$\partial_x u^\pm \mp \sqrt{m^2 - \lambda} u^\pm = f^\pm.$$

Then

$$\begin{cases} u^-(x) = \int_{-\infty}^x e^{-\sqrt{m^2-\lambda^2}(x-y)} f^-(y)\, dy, \\ u^+(x) = -\int_x^{+\infty} e^{\sqrt{m^2-\lambda^2}(x-y)} f^+(y)\, dy. \end{cases}$$

multiplying $u^\pm$ by the weight e^φ provides

$$\begin{cases} e^{\varphi(x)}|u^-(x)| \leq \int_{-\infty}^x e^{\varphi(x)-\varphi(y)-\sqrt{m^2-\lambda^2}(x-y)} |e^{\varphi(y)} f^-(y)|\, dy, \\ e^{\varphi(x)}|u^+(x)| \leq -\int_x^{+\infty} e^{\varphi(x)-\varphi(y)+\sqrt{m^2-\lambda^2}(x-y)} |e^{\varphi(y)} f^+(y)|\, dy, \end{cases}$$

since $\varphi(x) - \sqrt{m^2-\lambda^2}x \leq \varphi(y) - \sqrt{m^2-\lambda^2}y$ for $y \leq x$ and $\varphi(x) + \sqrt{m^2-\lambda^2}x \leq \varphi(y) + \sqrt{m^2-\lambda^2}y$ for $x \leq y$.

The rest of the proof goes the same way as before. □

Remark A.1. The above lemma can be localized outside any ball so that the assumptions can be just asserted only at infinity.

A.2 Hardy inequality in higher dimensions

Following Berthier and Georgescu [BG87], we have the following result.

Lemma A.5. *Let $n \geq 2$. If $\lambda \in \mathbb{C} \setminus \pm m$ and $s > -\frac{1}{2}$ then there is a constant $c = c(\lambda, s)$, locally bounded in λ and s, such that for every $u \in S'(\mathbb{R}^n, \mathbb{C}^N)$ having the property $\hat{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{C}^N)$ the following inequality is true;*

$$\|\langle Q\rangle^s u\|_{H^1} \leq c\|\langle Q\rangle^{s+1}(D_m - \lambda)u\|.$$

Proof. Notice that for $|\lambda| < m$ or $\lambda \notin \mathbb{R}$, the statement is immediate due to the invertibility of $D_m - \lambda$. For $\lambda \in (-\infty, -m) \cup (m, \infty)$, the lines of the proof follow the corresponding argument by Berthier and Georgescu. Since

$$\|D_0\langle Q\rangle^s u\| \leq \|[D_0, \langle Q\rangle^s]u\| + \|\langle Q\rangle^s D_0 u\| \leq 3|s|\|\langle Q\rangle^{s-1}u\| + \|\langle Q\rangle^s(D_m - \lambda)u\| + \|(m\beta - \lambda)\langle Q\rangle^s u\|,$$

it is enough to prove

$$\|\langle Q\rangle^s u\| \leq c\|\langle Q\rangle^{s+1}(D_m - \lambda)u\|.$$

Using the Fourier transform, it is equivalent to proving that for every $v \in S'(\mathbb{R}^n, \mathbb{C}^N)$ having the property $v \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{C}^N)$ one has

$$\|v\|_s \leq c\|(h_m(\xi) - \lambda)v\|_{s+1},$$

where $h_m(\xi)$ is the symbol of D_m. Then, up to a diagonalization, it is enough to prove that for each $u \in S'(\mathbb{R}^n, \mathbb{C}^N)$ with the property $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{C}^N)$ one has

$$\|v\|_s \leq c\|(\sqrt{\xi^2 + m^2} \pm \lambda)v\|_{s+1}.$$

Then one can either notice that the proof by Berthier and Georgescu will work, or one can recognize in the radial direction the symbol of the one-dimensional Dirac operator and use Lemma A.3 with $\varphi = (s + \frac{n-1}{2})\log\langle r\rangle$. □

We can also include the thresholds at some price:

Lemma A.6. *Let $n \geq 2$. If $\lambda \in \mathbb{C}$ and $s > -\frac{1}{2}$, then there is a constant $c = c(\lambda, s)$, locally bounded in λ and s, such that for every $u \in S'(\mathbb{R}^n, \mathbb{C}^N)$ having the property $\hat{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{C}^N)$ the following inequality is true:*

$$\|\langle Q\rangle^s u\|_{H^1} \leq c\|\langle Q\rangle^{s+2}(D_m - \lambda)u\|.$$

Diagonalizing the operator $\mathbf{J}$, one can immediately obtain the following result:

Lemma A.7. *If $\lambda \in \mathbb{C} \setminus \{\pm\mathrm{i}(m \pm \omega)\}$ and $s > -\frac{1}{2}$, then there is a constant $c = c(\lambda, \omega, s)$, locally bounded in ω, λ and s, such that for every $u \in S'(\mathbb{R}^n, \mathbb{C}^{2N})$ having the property $\hat{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{C}^{2N})$ the following inequality is true:*

$$\|\langle Q\rangle^s u\|_{H^1} \le c\|\langle Q\rangle^{s+1}\big(\mathbf{J}(\mathbf{D}_m - \omega) - \lambda\big)u\|.$$

If $\lambda \in \mathbb{C}$ and $s > -\frac{1}{2}$, then there is a constant $c = c(\lambda, \omega, s)$, locally bounded in ω, λ and s, such that for every $u \in S'(\mathbb{R}^n, \mathbb{C}^{2N})$ having the property $\hat{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{C}^{2N})$ the following inequality is true;

$$\|\langle Q\rangle^s u\|_{H^1} \le c\|\langle Q\rangle^{s+2}\big(J(D_m - \omega) - \lambda\big)u\|$$

B Appendix: Carleman-Berthier-Georgescu estimates

Lemma B.1 (Carleman-Berthier-Georgescu inequality, $n = 3$; Theorem 5 in [BG87]). *Let $n = 3$. For any $|\lambda| > m$, we have the following estimate for some $C, R > 0$:*

$$\forall\tau \ge 1, \qquad \sqrt{\tau}\|e^{\tau|x|}u\| + \|e^{\tau|x|}u\|_{L^{10/3}} \le C\|\sqrt{|x|}e^{\tau|x|}(D_m - \lambda)u\|, \qquad u \in H^1_c(\Omega_R, \mathbb{C}^N). \tag{B.1}$$

Above,

$$\Omega_R = \{x \in \mathbb{R}^n\ ;\ |x| > R\}.$$

Our aim is to have the following Carleman-Berthier-Georgescu inequalities for any dimension.
Let $D_m = -\mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla} + \beta m$, $\varphi \in C^1(\Omega)$, and denote $D^\varphi_m = e^\varphi D_m e^{-\varphi} = D_m + \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}\varphi$.

Lemma B.2 (Lemma 3, [BG87]). *For $v \in H^1_c(\Omega)$,*

$$\mathrm{Re}\langle(D_m - \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}\varphi + \lambda)v, (D^\varphi_m - \lambda)v\rangle = \|\boldsymbol{\nabla}v\|^2 + \langle v, [m^2 - \lambda^2 - (\boldsymbol{\nabla}\varphi)^2]v\rangle. \tag{B.2}$$

Let $r = |\boldsymbol{x}|$, $\hat{X} = \boldsymbol{x}\boldsymbol{\nabla}$, $\mathscr{D} = \frac{1}{2}\{\boldsymbol{x}, -\mathrm{i}\boldsymbol{\nabla}\} = -\mathrm{i}\hat{X} - \frac{in}{2}$.

Lemma B.3 (Lemma 4, [BG87]). *Let $0 \notin \Omega$, $\varphi \in C^2(\Omega)$. Then for any $u \in H^1_c(\Omega)$,*

$$\begin{aligned} &2\,\mathrm{Re}\langle(D_0 + 2\mathrm{i}\lambda\mathscr{D} + \{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\})v, (D^\varphi_m - \lambda)v\rangle \\ &\quad= 2\|\boldsymbol{\nabla}v\|^2 + 4\,\mathrm{Re}\langle\hat{X}v, \boldsymbol{\nabla}\varphi\boldsymbol{\nabla}v\rangle + 2\,\mathrm{Re}\langle\hat{X}v, \varDelta\varphi v\rangle + \langle v, [\hat{X}(\boldsymbol{\nabla}\varphi)^2]v\rangle. \end{aligned} \tag{B.3}$$

Proof. First, from $[\mathscr{D}, D_0] = [-\mathrm{i}\boldsymbol{x}\boldsymbol{\nabla}, -\mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}] = [\boldsymbol{\alpha}\boldsymbol{\nabla}, \boldsymbol{x}\boldsymbol{\nabla}] = \boldsymbol{\alpha}\boldsymbol{\nabla} = \mathrm{i}D_0$, we get

$$4\,\mathrm{Im}\langle\mathscr{D}v, (D^\varphi_m - \lambda)v\rangle = \frac{2}{\mathrm{i}}\langle v, [\mathscr{D}, D_0]v\rangle + 4\,\mathrm{Im}\langle\mathscr{D}v, (\beta m + \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}\varphi - \lambda)v\rangle = 2\langle v, D_0 v\rangle + 4\,\mathrm{Re}\langle\mathscr{D}v, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi v\rangle. \tag{B.4}$$

By using the identity

$$(\boldsymbol{\alpha}\boldsymbol{S})(\boldsymbol{\alpha}\boldsymbol{T}) = \boldsymbol{S}\boldsymbol{T} + \mathrm{i}\Sigma(\boldsymbol{S}, \boldsymbol{T}) := \boldsymbol{S}\boldsymbol{T} + \mathrm{i}\Sigma_{jk}S_j T_k, \qquad \boldsymbol{S},\ \boldsymbol{T} \in \mathbb{C}^n, \tag{B.5}$$

where the matrices $\Sigma_{jk} = \frac{1}{2\mathrm{i}}[\alpha^j, \alpha^k]$ are hermitian for each $j,\ k$, we have

$$D_0(D^\varphi_m - \lambda) = -\varDelta + mD_0\beta + \mathrm{i}D_0(\boldsymbol{\alpha}\boldsymbol{\nabla}\varphi) - \lambda D_0 = -\varDelta + mD_0\beta + \varDelta\varphi + \mathrm{i}\Sigma(\boldsymbol{\nabla}, \boldsymbol{\nabla}\varphi) - \lambda D_0.$$

But $\{D_0, \beta\} = 0$ and $\Sigma(\boldsymbol{\nabla}, \boldsymbol{\nabla}\varphi) = \Sigma_{jk}\partial_j \circ \varphi_k = \Sigma_{jk}\varphi_{jk} + \Sigma_{jk}\varphi_k\partial_j = -\Sigma_{jk}\varphi_j\partial_k = -\Sigma(\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla})$, hence

$$2\,\mathrm{Re}\langle D_0 v, (D^\varphi_m - \lambda)v\rangle = 2\|\boldsymbol{\nabla}v\|^2 + \langle v, \varDelta\varphi v\rangle - 2\mathrm{i}\langle v, \Sigma(\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla})v\rangle - 2\lambda\langle v, D_0 v\rangle. \tag{B.6}$$

Adding (B.4) and (B.6), we obtain

$$\begin{aligned} &2\,\mathrm{Re}\langle D_0 v, (D^\varphi_m - \lambda)v\rangle + 4\lambda\,\mathrm{Im}\langle\mathscr{D}v, (D^\varphi_m - \lambda)v\rangle \\ &\quad= 2\|\boldsymbol{\nabla}v\|^2 + \langle v, \big(\varDelta\varphi - 2\mathrm{i}\Sigma(\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla})\big)v\rangle + 4\lambda\,\mathrm{Re}\langle\mathscr{D}v, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi v\rangle. \end{aligned} \tag{B.7}$$

Recalling that $[\mathscr{D}, D_0] = \mathrm{i}D_0$, we derive the identity

$$\begin{aligned}\{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\}D_0 &= \{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi D_0\} + \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi[\mathscr{D}, D_0] = \{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi D_0\} + \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}\varphi D_0\\ &= \{\mathscr{D} + \frac{\mathrm{i}}{2}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi D_0\} = -\mathrm{i}\{\mathscr{D} + \frac{\mathrm{i}}{2}, \boldsymbol{\nabla}\varphi\boldsymbol{\nabla}\} + \{\mathscr{D} + \frac{\mathrm{i}}{2}, \Sigma(\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla})\},\end{aligned}$$

where in the last line we used (B.5). Due to $\mathrm{Re}\{\mathscr{D}, \Sigma(\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla})\} = 0$, the above relation leads to

$$2\,\mathrm{Re}\{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\}D_0 = \mathrm{Im}\{2\mathscr{D} + \mathrm{i}, \boldsymbol{\nabla}\varphi\boldsymbol{\nabla}\} + \mathrm{Re}\{2\mathscr{D} + \mathrm{i}, \Sigma(\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla})\} = \mathrm{Im}\{2\mathscr{D} + \mathrm{i}, \boldsymbol{\nabla}\varphi\boldsymbol{\nabla}\} + 2\mathrm{i}\Sigma(\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}). \quad \text{(B.8)}$$

The first term in the right-hand side of (B.8) is given by

$$\begin{aligned}\mathrm{Im}\{2\mathscr{D} + \mathrm{i}, \boldsymbol{\nabla}\varphi\boldsymbol{\nabla}\} &= -\mathrm{i}\Big(\mathscr{D}\boldsymbol{\nabla}\varphi\boldsymbol{\nabla} + \boldsymbol{\nabla} \circ \boldsymbol{\nabla}\varphi\mathscr{D} + \boldsymbol{\nabla} \circ \boldsymbol{\nabla}\varphi\mathscr{D} + \mathscr{D}\boldsymbol{\nabla}\varphi\boldsymbol{\nabla} + \mathrm{i}\boldsymbol{\nabla}\varphi\boldsymbol{\nabla} - \mathrm{i}\boldsymbol{\nabla} \circ \boldsymbol{\nabla}\varphi\Big)\\ &= -\mathrm{i}\{\mathscr{D}, \{\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}\}\} - \mathit{\Delta}\varphi = 2\,\mathrm{Im}(\mathscr{D}\{\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}\}) - \mathit{\Delta}\varphi = -2\,\mathrm{Re}(\boldsymbol{\nabla} \circ \boldsymbol{x}\{\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}\}) - \mathit{\Delta}\varphi\\ &= -4\,\mathrm{Re}(\boldsymbol{\nabla} \circ \boldsymbol{x}\,\boldsymbol{\nabla}\varphi\boldsymbol{\nabla}) - 2\,\mathrm{Re}(\boldsymbol{\nabla} \circ \boldsymbol{x}\mathit{\Delta}\varphi) - \mathit{\Delta}\varphi. \qquad \text{(B.9)}\end{aligned}$$

From (B.8) and (B.9) we obtain $2\,\mathrm{Re}\{\mathscr{D}, \boldsymbol{\alpha}\nabla\varphi\}D_0 = -4\,\mathrm{Re}(\boldsymbol{\nabla} \circ \boldsymbol{x}\,\boldsymbol{\nabla}\varphi\boldsymbol{\nabla}) - 2\,\mathrm{Re}(\boldsymbol{\nabla} \circ \boldsymbol{x}\mathit{\Delta}\varphi) - \mathit{\Delta}\varphi + 2\mathrm{i}\Sigma(\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla})$,

$$\begin{aligned}&2\,\mathrm{Re}\langle\{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\}v, (D_m^\varphi - \lambda)v\rangle\\ &\quad= 2\,\mathrm{Re}\langle\{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\}v, D_0 v\rangle + 2\,\mathrm{Re}\langle\{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\}v, \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}\varphi v\rangle - 2\lambda\,\mathrm{Re}\langle\{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\}v, v\rangle\\ &\quad= 4\,\mathrm{Re}\langle\hat{X}v, \boldsymbol{\nabla}\varphi\boldsymbol{\nabla}v\rangle + 2\,\mathrm{Re}\langle\hat{X}v, \mathit{\Delta}\varphi v\rangle - \langle v, (\mathit{\Delta}\varphi - 2\mathrm{i}\Sigma(\boldsymbol{\nabla}\varphi, \boldsymbol{\nabla}))v\rangle\\ &\qquad+\langle v, \hat{X}(\boldsymbol{\nabla}\varphi)^2 v\rangle - 4\lambda\,\mathrm{Re}\langle\mathscr{D}v, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi v\rangle. \qquad \text{(B.10)}\end{aligned}$$

Above, we used the identity

$$2\,\mathrm{Re}\langle\{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\}v, \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}\varphi v\rangle = \langle v, \big(\{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\}\mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}\varphi - \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\}\big)v\rangle = \langle v, \hat{X}(\boldsymbol{\nabla}\varphi)^2 v\rangle.$$

Adding (B.7) and (B.10) yields (B.3). □

Now we assume that φ is radial. Denote $\hat{\alpha} = r^{-1}\boldsymbol{\alpha}\boldsymbol{x}$. We subtract (B.2) from (B.3) with the aid of the identity

$$\{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\} = \{-\mathrm{i}\hat{X} - \frac{in}{2}, \hat{\alpha}\varphi'\} = -2\mathrm{i}\hat{\alpha}\varphi'\hat{X} - \mathrm{i}\hat{\alpha}r\varphi'' - \mathrm{i}n\hat{\alpha}\varphi',$$

arriving at

$$\begin{aligned}&\Big\langle\Big[D_0 + 2\mathrm{i}\lambda\mathscr{D} + \{\mathscr{D}, \boldsymbol{\alpha}\boldsymbol{\nabla}\varphi\} - \tfrac{1}{2}(D_m - \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}\varphi + \lambda)\Big]v, (D_m^\varphi - \lambda)v\Big\rangle\\ &= 2\,\mathrm{Re}\Big\langle\Big[-\tfrac{\mathrm{i}}{2}\boldsymbol{\alpha}\boldsymbol{\nabla} + 2(\lambda - \mathrm{i}\hat{\alpha}\varphi')\hat{X} + (n - \tfrac{1}{2})\lambda - \tfrac{m}{2}\beta - \mathrm{i}(n - \tfrac{1}{2})\hat{\alpha}\varphi' - \mathrm{i}\hat{\alpha}r\varphi''\Big]v, (D_m^\varphi - \lambda)v\Big\rangle\\ &= \|\boldsymbol{\nabla}v\|^2 + 4\langle\hat{X}v, \tfrac{\varphi'}{r}\hat{X}v\rangle + 2\,\mathrm{Re}\langle\hat{X}v, \mathit{\Delta}\varphi v\rangle + \Big\langle v, \Big[\lambda^2 - m^2 + \varphi'^2 + 2r\varphi'\varphi''\Big]v\Big\rangle. \qquad \text{(B.11)}\end{aligned}$$

Since

$$2\,\mathrm{Re}\langle\hat{X}v, \frac{\varphi'}{r}v\rangle = \langle\hat{X}v, \frac{\varphi'}{r}v\rangle + \langle v, \frac{\varphi'}{r}\hat{X}v\rangle = \langle v, (-\hat{X} \circ \frac{\varphi'}{r} - n\frac{\varphi'}{r} + \frac{\varphi'}{r}\hat{X})v\rangle = -\langle v, \Big[\varphi'' v + (n-1)\frac{\varphi'}{r}\Big]v\rangle,$$

which is valid for $v \in H_c^1(\mathit{\Omega})$ with $\mathit{\Omega} \not\ni 0$, we have

$$2\,\mathrm{Re}\langle\hat{X}v, \mathit{\Delta}\varphi v\rangle = 2\,\mathrm{Re}\langle\hat{X}v, \varphi'' v\rangle - \Big\langle v, \Big[(n-1)\varphi'' + (n-1)^2\frac{\varphi'}{r}\Big]v\Big\rangle.$$

We use the above relation to rewrite (B.11) as

$$\begin{aligned}&\mathrm{Re}\Big\langle\Big[-\mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla} + 4(\lambda - \mathrm{i}\hat{\alpha}\varphi')\hat{X} + (2n-1)(\lambda - \mathrm{i}\hat{\alpha}\varphi') - m\beta - 2\mathrm{i}\hat{\alpha}r\varphi''\Big]v, (D_m^\varphi - \lambda)v\Big\rangle\\ &= \|\boldsymbol{\nabla}v\|^2 + 4\|(\tfrac{\varphi'}{r})^{\frac{1}{2}}\hat{X}v\| + 2\,\mathrm{Re}\langle\hat{X}v, \varphi'' v\rangle + \Big\langle v, \Big[\lambda^2 - m^2 + \varphi'^2 + 2r\varphi'\varphi'' - (n-1)\varphi'' - (n-1)^2\tfrac{\varphi'}{r}\Big]v\Big\rangle.\end{aligned}$$

For any $\delta > 0$ and any $\mathscr{M}(r) > 0$, the above relation yields the following inequality:

$$\begin{aligned}
&\Big\| \big[- \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla} + 4(\lambda - \mathrm{i}\hat{\alpha}\varphi')\hat{X} + (2n-1)(\lambda - \mathrm{i}\hat{\alpha}\varphi') - m\beta - 2\mathrm{i}\hat{\alpha}r\varphi'' \big] \frac{v}{2\mathscr{M}^{1/2}} \Big\|^2 + \|\mathscr{M}^{1/2}(D_m^\varphi - \lambda)v\|^2 \\
&\qquad \ge \|\boldsymbol{\nabla} v\|^2 + (4-\delta)\|(\frac{\varphi'}{r})^{1/2}\hat{X}v\|^2 - \delta^{-1}\|(\frac{r}{\varphi'})^{1/2}\varphi'' v\|^2 \\
&\qquad\qquad + \Big\langle v, \Big[\lambda^2 - m^2 + \varphi'^2 + 2r\varphi'\varphi'' - (n-1)\varphi'' - (n-1)^2\frac{\varphi'}{r}\Big] v \Big\rangle.
\end{aligned} \tag{B.12}$$

To eliminate from (B.12) the terms with $\boldsymbol{\nabla} v$ and $\hat{X}v$, we need to take $\delta > 0$ and $\mathscr{M}(r)$ so that

$$\|\frac{1}{2\mathscr{M}^{1/2}}\big[- \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla} + 4(\lambda - \mathrm{i}\hat{\alpha}\varphi')\hat{X}\big]u\|^2 \le \|\boldsymbol{\nabla} v\|^2 + (4-\delta)\|(\frac{\varphi'}{r})^{1/2}\hat{X}v\|^2,$$

which would follow from

$$\frac{n}{4\mathscr{M}} \le 1, \qquad 4\frac{\lambda^2 + \varphi'^2}{\mathscr{M}} \le (4-\delta)\frac{\varphi'}{r}.$$

It is enough to take

$$\delta = 2, \qquad \mathscr{M}(r) = \frac{n}{4} + 2\frac{\lambda^2 r}{\varphi'} + 2r\varphi'. \tag{B.13}$$

Now (B.12) takes the form

$$\begin{aligned}
&\|\frac{1}{2\mathscr{M}^{1/2}}\Big[(2n-1)\lambda - m\beta - \mathrm{i}(2n-1)\hat{\alpha}\varphi' - 2\mathrm{i}\hat{\alpha}r\varphi''\Big]v\|^2 + \|\mathscr{M}^{1/2}(D_m^\varphi - \lambda)v\|^2 \\
&\qquad \ge \Big\langle v, \Big[\lambda^2 - m^2 + \varphi'^2 + 2r\varphi'\varphi'' - (n-1)\varphi'' - (n-1)^2\frac{\varphi'}{r} - \frac{r\varphi''^2}{2\varphi'}\Big] v \Big\rangle,
\end{aligned} \tag{B.14}$$

which we rewrite as

$$\begin{aligned}
\|\mathscr{M}^{1/2}(D_m^\varphi - \lambda)v\|^2 \ge \Big\langle v, \Big[\lambda^2 - m^2 + \varphi'^2 + 2r\varphi'\varphi'' \\
-(n-1)\varphi'' - (n-1)^2\frac{\varphi'}{r} - \frac{r\varphi''^2}{2\varphi'} - \frac{(2n-1)(|\lambda| + \varphi') + m + 2r|\varphi''|}{4\mathscr{M}(r)}\Big] v \Big\rangle.
\end{aligned} \tag{B.15}$$

Lemma B.4 (Carleman-Berthier-Georgescu inequality in $\mathbb{R}^n$)**.** *Let $\lambda \in \mathbb{R}$. Let $\varphi \in C^2(\mathbb{R}_+)$ with $\varphi' > 0$. Assume that there is $R_\varphi > 0$ such that for all $r \ge R_\varphi$ one has*

$$\lambda^2 - m^2 + \frac{\varphi'^2}{2} > 0, \tag{B.16}$$

$$\frac{1}{2}\varphi'^2 + 2r\varphi'\varphi'' \ge (n-1)\varphi'' + (n-1)^2\frac{\varphi'}{r} + \frac{r\varphi''^2}{2\varphi'} + \frac{3n}{8r} + \frac{m}{8r\varphi'} + \frac{|\varphi''|}{4\varphi'}. \tag{B.17}$$

Then

$$\|\Big(\lambda^2 - m^2 + \frac{\varphi'^2}{2}\Big)^{1/2} e^\varphi v\| \le \|\big(\frac{n}{4} + 2\frac{\lambda^2 r}{\varphi'} + 2r\varphi'\big)^{1/2} e^\varphi (D_m - \lambda)v\|, \qquad v \in H_c^1(\Omega_{R_\varphi}).$$

Moreover for any $v \in H_0^1(\Omega_{R_\varphi})$ such that $\big(\frac{n}{4} + 2\frac{\lambda^2 r}{\varphi'} + 2r\varphi'\big)^{1/2} e^\varphi (D_m - \lambda)v \in L^2(\mathbb{R}^n, \mathbb{C}^N)$

$$\|\Big(\lambda^2 - m^2 + \frac{\varphi'^2}{2}\Big)^{1/2} e^\varphi v\| \le \|\big(\frac{n}{4} + 2\frac{\lambda^2 r}{\varphi'} + 2r\varphi'\big)^{1/2} e^\varphi (D_m - \lambda)v\|.$$

Proof. One has $\frac{(2n-1)|\lambda|}{n + 8\frac{\lambda^2 r}{\varphi'} + 8r\varphi'} \le \frac{(2n-1)|\lambda|}{16r|\lambda|} + \frac{m}{8r\varphi'} \le \frac{n}{8r}$, hence the inequality (B.17) implies that, as long as $r \ge R_\varphi$, the following relation holds:

$$\frac{1}{2}\varphi'^2 + 2r\varphi'\varphi'' \ge (n-1)\varphi'' + (n-1)^2\frac{\varphi'}{r} + \frac{r\varphi''^2}{2\varphi'} + \frac{(2n-1)(|\lambda| + \varphi') + m + 2r|\varphi''|}{n + 8\frac{\lambda^2 r}{\varphi'} + 8r\varphi'}. \tag{B.18}$$

Now we use the inequality (B.15) (with $\mathscr{M}$ given by (B.13)) together with (B.18), substitute $e^\varphi v$ in place of v, and use the identity $D_m^\varphi(e^\varphi v) = e^\varphi D_m v$, arriving at the desired inequality.

The extension to non compactly supported functions follows exactly the process of the proof of Lemma A.3. □

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